DOCUMENT RESUME

ED 143 552 SE 023 036

AUTHOR	Maletsky, Evan M.; And Others
TITLE	Studies in Mathematics, Volume XII. A Brief Course in Mathematics for Junior High School Teachers. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	66
NOTE	400p.; For related documents, see SE 023 028-041; Contains numerous light type
EDRS PRICE	MF-$0.83 HC-$20.75 Plus Postage.
DESCRIPTORS	*Arithmetic; *Geometry; Inservice Education; Junior High Schools; Measurement; *Number Concepts; Secondary Education; *Secondary School Mathematics; Statistics; *Teaching Guides
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This text was written under the auspices of the School Mathematics Study Group (SMSG) as a means for preparing teachers to teach the SMSG text Mathematics for Junior High School, Volume I. Included throughout are comments on suggested methods of presenting material to seventh graders. In this text, class exercises are interspersed throughout, with answers at the conclusion of each chapter. Answers to the exercises at the end of each chapter are found at the end of the book. The text was written with the thought that an instructor would be available, though sufficient details have been presented so that a teacher should be able to master the material independently. The material in the book should help any teacher teach a modern approach to mathematics that deals with: (1) an emphasis on the rationale of the fundamental operations; (2) a discussion of properties and structure of the number system; (3) attention to concepts of non-metric as well as metric geometry; and (4) exploration of other systems of numeration as a device for strengthening the understanding of our own decimal system. (Author/RH)

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MATHEMATICS
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STUDIES IN MATHEMATICS

VOLUME XII

A Brief Course in Mathematics For Junior High School Teachers

(revised edition)

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STUDIES IN MATHEMATICS

Volume XII

A BRIEF COURSE IN MATHEMATICS FOR JUNIOR HIGH SCHOOL TEACHERS

(revised edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

This text has been written under the auspices of the School Mathematics Study Group as a means of preparing one to teach the SMSG text *Mathematics for Junior High School*, Volume I. It attempts to develop the basic content necessary to understand and teach the material covered in Volume I of the junior high school series.

Also included throughout this text are comments on suggested methods of presenting this material to seventh graders. Additional helpful hints can be found in the SMSG Teacher's Commentary that accompanies *Mathematics for Junior High School*, Volume I. Thus, it would be quite beneficial for one to study this text concurrently with the available SMSG seventh grade materials.

Although designed specifically to accompany the aforementioned SMSG text, the material presented herein should adequately prepare one to teach any of the so-called "modern" approaches to seventh grade mathematics. Almost all of these programs have certain key features in common, such as:

(a) emphasis on the rationale of the fundamental operations;

(b) discussion of properties and structure of the number system;

(c) attention to concepts of non-metric as well as metric geometry;

(d) exploration of other systems of numeration as a device for strengthening the understanding of our own decimal system.

It has been the experience of teachers who have participated in such programs as the SMSG one that seventh grade youngsters (as well as teachers) show far more interest and enthusiasm in their studies of mathematics than is the case with traditional programs that present a heavy emphasis on computational techniques. This is not to imply that computation is neglected in the newer approaches; rather it is developed with careful attention paid to meaning and understanding.

In this text, class exercises are interspersed throughout, with answers given at the conclusion of each chapter. Answers to the end of chapter exercises are to be found at the end of the book. The exercises should be completed as soon as the material is read in order to strengthen ideas presented within each section. Furthermore, each chapter closes with an additional collection of exercises to provide practice of key ideas. A series of masters are available for preparing projectuals to use in conjunction with the teaching of a course based on this book.

This text was written with the thought that it would be used in an in-service course for which there would be an instructor or consultant available. However, sufficient details have been presented throughout so that a teacher should be able to master the material independently.

Although these units are based on Volume I of Mathematics for Junior High School, it was necessary to present some ideas that first appear in Volume II in order to provide a complete picture in some areas. Thus, the set of real numbers is discussed here although they are not formally treated until the eighth grade in most texts.

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INTRODUCTION

Mathematics is concerned with many things: some serious and some frivolous, some hard and some easy. The computation of batting averages is mathematics. The study of surfaces which may be pulled into the shape of spheres is mathematics. The solution of the well-known problem,

$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

where each letter is to be replaced uniquely by a digit to form a correct addition problem, is mathematics. Some very simple but careful reasoning will produce the answer. In this text we shall study some of the branches of mathematics and lay a foundation for further study.

The diversity in mathematics may be compared to the diversity available in reading. We may choose our reading in many ways. We may read for recreation or for knowledge. This analogy may be carried on in other ways. Just as some read on rare occasions, others may read compulsively, being uncomfortable when they are more than three feet from a book, paper, or magazine. Whatever the reason or whatever the level, nearly everyone finds something of interest to read. Certainly, everyone who can read finds the ability to read valuable. The same may be said for mathematics: There is something of interest or value to everyone. There are those who will use mathematics to verify their paychecks and there are some who are compulsive mathematicians, only happy when thinking of mathematics.

The analogy may be extended in still other directions. No one person is able to read all the books, magazines, pamphlets, and papers published, and no one individual can be knowledgeable in all areas of mathematics. There are those who read and also write; in mathematics there are those who study, those who use mathematics, and those who go further and create new mathematics.

Every discipline has a vocabulary of its own. This may include special words, such as magnetohydrodynamics, or common words with meanings specialized to the subject, such as function. This is usually an attempt to achieve precision and economy in communication. Unfortunately a jargon is sometimes introduced in a discipline to mask a fundamental lack of knowledge and to appear scientific.

To achieve economy and precision in mathematics some very common words are used to convey deep ideas. The word "number," as used in mathematics, briefly conveys a very abstract idea. To the layman, the word "number" brings to mind some symbols. We recognize that the symbol is not the concept. When we write the word "horse" we think of a "solid-hoofed animal used for riding on or drawing burdens," but the word "horse" is not the animal but is rather a symbol for the animal. When we write the symbols 6, VI, and $\frac{24}{4}$, we are writing various forms of names for a number concept. To be precise we speak of the symbols 6, VI, and $\frac{24}{4}$ as numerals which name the same number. There are, of course, situations in which this degree of precision is necessary and other occasions when this precision becomes pedantic. A physician, when speaking to a colleague, may refer to a patient's broken tibia while the patient is content to speak of a broken leg.

We might speak of the numbers whose numerals are 1, 2, 3, and 4. This would be perfectly correct. However, we frequently choose to write "the numbers 1, 2, 3, and 4" and trust the context will make our meaning clear.

It is desirable that pupils know the distinction between numeral and number. For example, we may wish to write:

$$\frac{4[(3 \times 3) + 3]}{6} = 8 .$$

On the face of it, it is ridiculous to claim the two sides of this expression are the same. When we realize these are two names which name the same number the statement becomes meaningful.

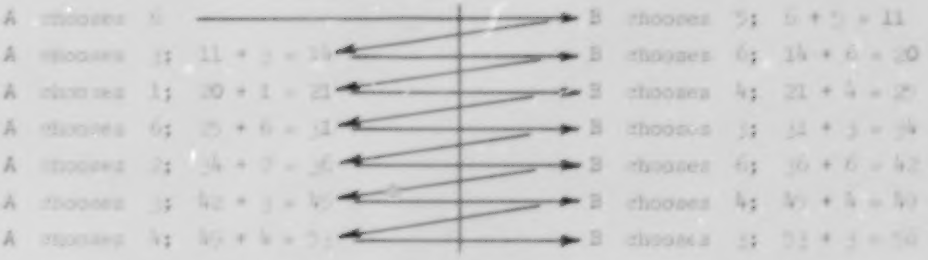
The purpose of the introduction in this text is to illustrate that there is more to mathematics than computing and that there is much of interest in mathematics that may readily be demonstrated to the junior high school student. The aims of an introductory chapter at the beginning of the seventh grade are many. It is worthwhile to reawaken an interest in mathematics, to show the power of mathematics in many varied situations, and to give an indication of some of the things to come.

Chapter 1 in Volume I of Mathematics for Junior High School illustrates how this might be done. In teaching such a chapter it may not be best to do the entire chapter at one time. A part of such a chapter could be covered until the student's interest is captured and the rest postponed until later. A good time to return to such material is just before vacations.

A Number Game

The interest of the student might be aroused by playing a very simple number game. Here is a game with simple rules for two players. To describe the game we will call the players A and B. From among the numbers 1, 2, 3, 4, 5, 6 player A picks a number. Player B then picks a number, again from 1, 2, 3, 4, 5, 6 and adds it to the number A picked. It is now A's turn. He picks a number from the six and adds it to the preceding sum. The game continues in this fashion. The game is won by the player who is able on his turn to pick the number, from 1, 2, 3, 4, 5, 6, which makes the total sum 56. The same number may be picked as many times as desired.

A Sample Game



A chooses 6	B chooses 5; 6 + 5 = 11
A chooses 3; 11 + 3 = 14	B chooses 6; 14 + 6 = 20
A chooses 1; 20 + 1 = 21	B chooses 4; 21 + 4 = 25
A chooses 6; 25 + 6 = 31	B chooses 3; 31 + 3 = 34
A chooses 2; 34 + 2 = 36	B chooses 6; 36 + 6 = 42
A chooses 3; 42 + 3 = 45	B chooses 4; 45 + 4 = 49
A chooses 4; 49 + 4 = 53	B chooses 3; 53 + 3 = 56

B wins!

Pair off the class members and play this game. Can you find a pattern that will enable you to always win this game?

Is this mathematics? We would say it is, for reason and deduction allows you to answer the above questions. This game has a feature that is desirable in the classroom; it may be varied. For example, the game may be played with the numbers 1, 2, 3, 4, 5, 6, 7 and winning sum 85. There are, of course, many variations.

This game is an example of a mathematical problem (or puzzle) which may be solved without any formal knowledge of mathematics. It is amusing to play and it is pleasurable to discover the strategy of the game. Since there are variations in which the winning strategy is not much changed, many students have an opportunity to make a discovery.

In the remainder of this introduction we shall examine in detail some problems which are typical of those given in Chapter 1 of Mathematics for Junior High School, Volume I. There are many such problems; their object is to increase interest in mathematics. They should not be allowed to become frustrating. A simple question at the right time may lead the student or class in the right direction.

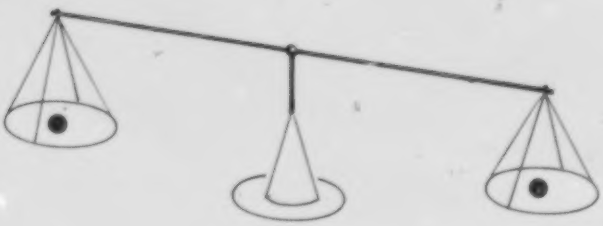
Weighing Problems

Consider the following reasoning problem:

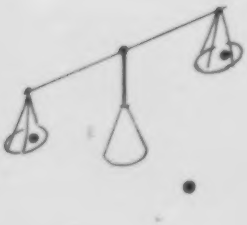
> Eight marbles all have the same size, color, and shape. Seven of them have the same weight and the other is heavier. Using a balance scale, how would you find the heavy marble if you make only two weighings?

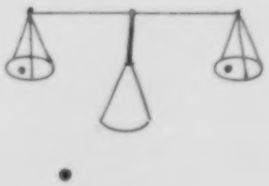
There are many ways one might begin to work on this problem. Many students, however, hesitate to begin because of the relatively large number of marbles involved. This difficulty occurs in many problems and students should be encouraged to invent similar but easier problems to be solved first. The eight-marble problem is a good example of a problem in which this may easily be done.

A student might begin with this problem: Two marbles are identical in appearance. One marble is heavier than the other. Determine which marble is the heavy one using a beam balance. The solution of this problem is of course very easy.



Now one might be encouraged to look at the problem of three marbles. Is it possible to determine the heavy marble with one weighing? The student should now come to the conclusion that an equal number of marbles must be placed in each pan (tray) of the beam balance. He also will see that there is a need to consider cases.





While it is not recommended that the students work their way case by case up to the case of eight marbles, a few more cases may be helpful. Can the heavy marble among four marbles always be determined in one weighing? In two weighings? Having done the problem for four marbles, the problem of five marbles is easily done.

Studying the cases with a smaller number of marbles has served the purpose of making the problem seem less formidable and gives suggestions for doing the problem of eight marbles. We must be careful, however, that our special cases do not mislead us. It may appear for the cases already done that there is a different way to start in the case of an even number of marbles and an odd number of marbles. To do the problem of eight marbles, it may seem that the first weighing should be four marbles against four marbles. This would surely tell us which collection of four marbles contains the heavy marble. We know from the exploratory problems that it takes two weighings to determine a heavy marble from among four marbles. This approach to the problem would thus require three weighings. If it is possible to do the problem as stated, this is not the way to do it!

The elementary distinction between even and odd which may seem to indicate a general pattern does not in this case reveal the true pattern. It may happen as it did here, that a small number of cases seem to indicate a pattern which turns out not to be the correct pattern. If you have safely passed cars on a certain hill three times, can you make any valid assertions about what will happen the fourth time? Of course, the true pattern may be discovered with a small number of experiments.

To continue the solution, we do know that if the heavy marble is known to be among three marbles or among two marbles it may be located with one weighing. These remarks should allow the reader to complete the problem.

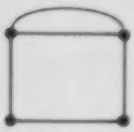
It is by no means implied that this problem must be worked by this sequence of discovery steps. What has been shown is that a seemingly complex problem may sometimes be done by exploration through simple problems. It is important for students ultimately to produce correct answers and equally important that they not be forced into the same mode of thought as their teachers.

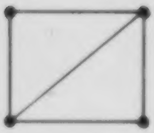
This class of problems was proposed originally as counterfeit coin problems; an example is given below:

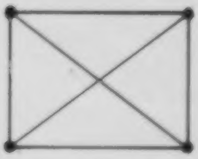
Among six coins identical in appearance there is one counterfeit coin. It is known that the counterfeit is made from impure metal and does not weigh the same as the genuine coins. What is the smallest number of weighings with a beam balance which would be required to locate the counterfeit coin? Will the answer change with additional requirements? Are additional weighings required to determine whether the false coin is too heavy or too light?

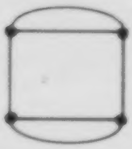
Unicursal Problems

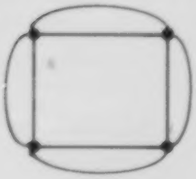
Most children have worked at problems which mathematicians call Unicursal problems. A figure is given composed of segments, either line segments or curves, and the player is required to trace the figure without lifting his pencil and without retracing a segment which has already been covered. Try this with the following figures.





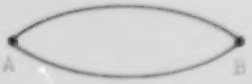




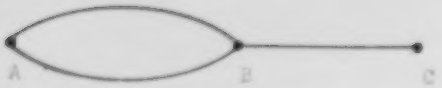


Anyone who has attempted one of the puzzles will be surprised to learn that the key to understanding them is mathematics. Again, no involved mathematics is required. Unicursal problems are another good illustration of the power and versatility of mathematics.

Again, let us begin by examining some simple problems. Let us see how a student might be encouraged to make discoveries.

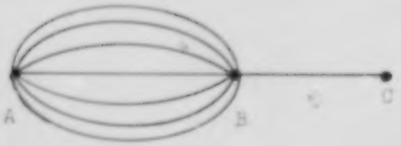


The figure above is easily traced according to the rules. Also the figure

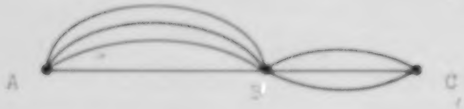


may be traced without violating the rules. In this case a little care must be exercised. A tracing which obeys the rules of the game, or what we will call a successful tracing, cannot begin at A but must begin at B or C. The student should put into words a reason for this statement. That is, why will it be impossible to complete a successful tracing if begun at A?

The figure below may also be traced according to the rules.



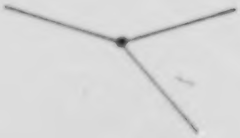
Why can't a successful tracing begin at B ? Why must the tracing begin or end at A or begin or end at C ?



Is it possible to trace the figure above? If it is possible, may you start at any point?

We would like to make some general statements about these problems, if possible. The rule which says the pencil is not to be lifted from the paper tells us that a figure composed of two disjoint parts cannot be traced successfully. To use a technical word: Any figure which can be traced according to the rules must be connected. In the last four figures, do the answers concerning tracing depend on the number of segments meeting at a point?

The examples indicate that the solution may not depend upon the total number of segments at a point; rather, the examples indicate a difference according to the parity of the number of segments at a point. Parity refers to the property of being odd or even. The examples suggest that a successful tracing of a figure with an odd number of segments at a point will begin or end at that point. For a point with one segment (part of a larger figure) it is clear that the tracing must start or end at that point. Let us think about a point with three segments.



(part of a larger figure)

Suppose we do not begin at this point, then in the course of a tracing which is to be successful we must come into this point, thereby covering one of the segments. There remain two segments not yet covered. A successful tracing must continue by leaving the point over one of the segments. There remains one segment to be traced. Once the third segment is traced we are finished, for there is no way then to move away from the point. That is, a point with three segments which is not a starting point of a successful tracing must be the finishing point.

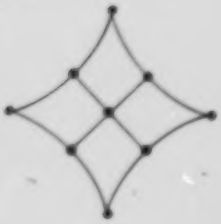
This reasoning may easily be extended to cover the case of any point with an odd number of segments. A point with an odd number of segments which is not a starting point of a successful tracing must be the end point; the tracing must stop at the point.

Let us review this conclusion. We look at an odd point, a point with an odd number of segments, and conclude that if it is not a starting point of a successful tracing then it is the end point. This does not eliminate the possibility that an odd point may be a starting point. We may say that an odd point which is not an end point is a starting point. We may classify the points of a successful tracing as: starting point, intermediate points, and end point. An odd point which is not the starting point must be the end point. An odd point can not be an intermediate point. Thus an odd point which is not the end point must be the starting point.

In a possible tracing problem each odd point must be a starting point or end point. Since, according to the rules, there can be at most one starting point and one end point, a figure which has more than two odd points cannot be traced successfully.

Is it possible to trace these figures without lifting your pencil and without retracing any segments?

(a).



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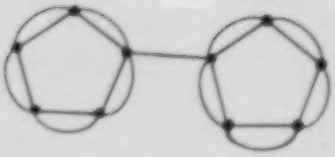

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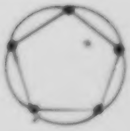
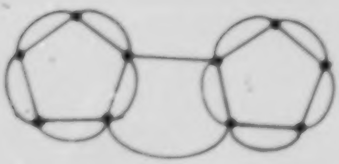
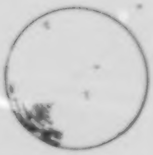


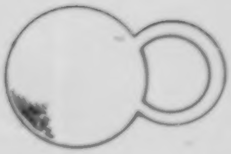
(d).



Unicursal problems, or tracing problems, have an honorable place in mathematics. The mathematician, Euler, (1707-1783) was the first to systematically study these puzzles in connection with the Koenigsberg bridge problem. Euler was a prolific mathematician; his collected works are still being edited. It is estimated that over sixty large volumes will be required. Among his many interests were the properties of figures in space.



sphere



sphere with a handle

For example, Euler was the first to give a mathematical way of differentiating between these two figures without saying "sphere with a handle". Euler's solving of this problem and the Unicursal problem was a part of the beginning of that branch of mathematics known as topology.

Euler wrote many mathematical texts. It has been claimed that until the recent flurry of the new texts, every high school mathematics text was a revision of Euler's texts.

Counting Problems

In a school class of 35 pupils, all the pupils take either French or German; 21 students are enrolled in French and 17 are enrolled in German. How many students are enrolled in both French and German?

This is an example of one form of counting problem. Let us analyze this problem. There must be some students enrolled in both French and German, for otherwise there would have to be 38 = 21 + 17 students in the class. When we add 21 and 17 we are counting among the 21 those students who are taking only French and those who are taking *both* French and German. Among the 17 we again count those students who are taking both French and German. The sum, 38 = 17 + 21, represents the number of those students taking only French, those students taking only German, and twice the number of students taking both French and German. The number 35 is the sum of the number of those taking *only* French, these taking *only* German; and the number taking both French and German.

$$\text{French} + \text{German} + \text{Both} = 35$$

$$\text{French} + \text{German} + 2 \times \text{Both} = 38$$

Now we see that the number taking both languages must be 3. From this we may compute, if we wish, the number taking only French, the number taking only German.

As a somewhat more complicated example of the same sort of problem we have:

> In a class of 225 students, all of whom are required to take a foreign language, 94 students are enrolled in French, 128 students are enrolled in German, and 88 are taking Russian. No other foreign languages are offered. It is also known that 36 are enrolled in French and German, 22 in German and Russian, and 29 in French and Russian. Are any students taking three languages? If so, how many?

The similarity between this problem and the preceding ones is clear. The similarity may suggest that we begin the problem as before. If we add 94, 128, and 88 to obtain 310, we find we have accounted for the students who are taking two languages twice. That is, a student who is taking French and Russian has been counted in the 94 students taking French and again in the 88 students taking Russian. Thus, it is not surprising that the sum, 310, exceeds the total number of students, 225. Suppose we attempt to represent the total number of students in terms of the number taking the various languages.

Thus, from 310 = 94 + 125 + 55 we must subtract the duplications. With some care the student may complete the solution in this fashion.

Caution: Subtraction to eliminate the duplication may result in other duplications. (There are exactly [illegible] students taking all three languages.)

In the next chapters we will see how, with the aid of a little mathematical notation and knowledge, the reasoning needed to do this problem may be much simplified.

Some of the many facets of mathematics have been introduced in the problems we have discussed. As examples of some other aspects of mathematics to come we will list some problems which you may think about and even solve now but whose solutions will be natural consequences of the material to be studied later. Answers to these and the other problems presented in this Introduction are included at the end of the text with the chapter exercise answers.

> There are three houses on a street. At the curb there are three utilities; water, electricity, and gas. Is it possible to connect each utility to each house without the connections crossing each other?
>
> Objects are to be weighed on a balance scale by comparing them with standard weights. If you wish to weigh objects, in pounds, between 1 pound and 63 pounds, what would be the most efficient set of standard weights? (Efficient means the smallest possible number of weights.)

Chapter 1

SETS

Introduction

Several questions usually arise among mathematicians, educators, pupils and parents about the pedagogical soundness of the teaching of sets and set language. Questions are raised as to why, where, when, and how sets should be introduced in the seventh grade curriculum. Some argue that a separate chapter should be included; some say that concepts of sets should be introduced as they are needed; and some educators claim that sets are not needed at all to be "modern."

There is merit in each of these viewpoints, but in this book we will take the position that for the junior high school youngsters set language should be presented primarily as it is needed to clarify mathematical concepts. The reason that we are including these concepts in a separate chapter in this text is that, because of the limited time a teacher has available to spend on an in-service program of this nature, familiarity with set language will expedite our presentation of other mathematical ideas appearing in later chapters. The language of sets will give us a precise way of talking about certain number ideas, properties of operations, and geometrical concepts.

1.1 The Concept of Sets

We say that a set is a well-defined collection of objects. What is meant by this? Certainly we know what is meant by a collection: A bunch of bananas, a herd of elephants, a set of dishes, the things on my desk, and so on. When we say it is well-defined, we must be certain that the description allows us to determine without ambiguity whether or not an element belongs to the set. The objects in a set need not be related in any way except that we treat them as a single group. For example, the set consisting of the number 5, the word "Tuesday," and the moon is a well-defined collection. However, in mathematics we usually speak of sets with elements that have some property in common. For example, the set of whole numbers, the set of primes, or the set of points on a line.

There are many ways of describing a set. For example, each of the following describe the same set:

The set of whole numbers between 6 and 12.
The set of whole numbers from 7 through 11, inclusive.
The numbers 7, 8, 9, 10, and 11.
$\{7, 8, 9, 10, 11\}$
$\{8, 10, 11, 9, 7\}$

Notice the use of braces, { }, with the *members* or *elements* of the set included between them. Frequently, an arbitrary capital letter is used to name the set.

$$M = \{7, 8, 9, 10, 11\}$$

The "things" in a set need not be objects you can touch or see. The set of all Beethoven symphonies does not contain any concrete objects. You may have heard some of its members, however. The set of all football teams in the United States is a set whose members are themselves sets of players.

Sometimes the symbol "ε" (stylized Greek letter, epsilon) is used to mean "is a member of," or "is an element of." Thus we can express the fact that the number 8 is a member of set M above by writing:

$$8 \in M .$$

We can express the fact that the number 6 is *not* a member of set M by writing

$$6 \notin M .$$

At times we encounter a set which contains no members. Such a set is called the "null set" or the "empty set," and is designated by { } or ∅. If set B is the set of all odd whole numbers less than 1, then set B has no members and we can write $B = \emptyset$. Another example of the empty set is the set of United States cities located in the province of Manitoba, Canada.

Often it is inconvenient to list all the members of a set within braces. The set of letters of the English alphabet could be shown as $E = \{a,b,c,\ldots,z\}$. Here a pattern has been established and the three dots mean "and so on in like manner" to z. The set of whole numbers may be shown as $W = \{0,1,2,3,4,\ldots\}$. The fact that no element is named after the ellipsis (...) implies that the listing of elements does not terminate but rather continues on in the same pattern without end. Such a set is called an *infinite* set. A *finite* set is a set which may be counted with the counting coming to an end. Set E above is an example of a finite set while set W illustrates an infinite set.

Some other examples of finite sets are:

$P = \{2,4,6,8,10\}$;
$Q = \{3,6,9,\ldots,81\}$;
the empty set, $\emptyset$;
the set of people in the United States of America.

Some additional infinite sets are:

$T = \{5,10,15,\ldots\}$;
the multiples of 257;
the points on a line;
the set of prime numbers.

Class Exercises

1. Tell whether or not each of the following sets is well-defined.
   a. The set of states of the United States bordering the Pacific Ocean.
   b. The set of small states in the United States.
   c. The set of all whole numbers which are not multiples of 3.
   d. The set of all whole numbers between 0 and 1.
   e. The letters which are in the name of your school and not in your last name.

2. Describe each of the following sets in at least two other ways:
   a. All odd whole numbers from 4 to 12 inclusive.
   b. $M = \{10,20,30,\ldots,100\}$.
   c. The set of integers greater than 50.
   d. The set of whole numbers between 20 and 30 and greater than 50.

3. Tell whether or not each of the following is true of false and explain your reasoning.
   a. $3 \in \{2,3,4,5\}$
   b. $\{0\} = \emptyset$
   c. $\{\emptyset\} = \emptyset$
   d. $17 \notin \{5,6,7,8,\ldots\}$
   e. $\{e,f,d\} \neq \{f,e,d\}$
   f. $32 \notin \{4,8,12,\ldots,96\}$

4. Classify the following sets as finite or infinite.

   a. Set of all whole numbers which are multiples of 3.
   b. Set of all numbers x such that $x + 1 = x$.
   c. Set of grains of sand on the beach of Coney Island.
   d. Set of all positive integers smaller than 0.
   e. All mathematics textbooks in the United States.

5. Let $M = \{1,3,7,\ldots,25\}$. What are the elements of this set? (Beware.)

1.2 Relations Between Sets

Consider the set of the first three letters of the alphabet, $A = \{a,b,c\}$, and the set containing the letters of the word cab, $B = \{c,a,b\}$. Since the order in which the members of a set are listed is immaterial, we can say that these two sets are identical or equal. This can be writtep as $A = B$. (Remember "=" here means "names precisely the same thing.")

Think of the sets $A = \{a,b,c,\ldots,z\}$ and $C = \{1,2,3,\ldots,26\}$. A matching or one-to-one correspondence may be illustrated between these two sets as follows:

$$\begin{array}{l} A = \{a,\ b,\ c,\ \ldots,\ z\} \\ \qquad\ \ \updownarrow\ \ \updownarrow\ \ \updownarrow \qquad\quad \updownarrow \\ C = \{1,\ 2,\ 3,\ \ldots,\ 26\} \end{array} .$$

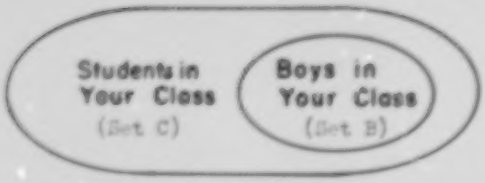
A one-to-one correspondence associates each element of set A to an element of set C and each element of set C to an element of set A. Obviously, other matchings are possible with the same two sets.

Certainly set A is not equal to set C, $A \neq C$, since they do not have the same elements. However, they do have the same cardinality; that is, the same number of members. Therefore, we say that set A is equivalent to set C. The equivalence of two sets is frequently written as: $A \leftrightarrow C$. Remember, two sets are equivalent if the elements of each can be put in a one-to-one correspondence.

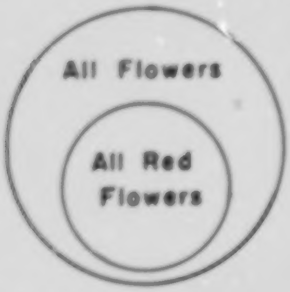
It should be apparent from the definitions that all equal sets are equivalent, but not all equivalent sets are equal.

If two sets have no members in common, we say they are disjoint. For example, consider the sets $R = \{6,8,12,14\}$ and $S = \{5,7,9\}$. Note that R and S have no common members. Therefore, we say that R and S are disjoint sets.

Think of the set of members of your class, C. The set of boys in your class, B, is a subset of the set of members of your class. This may be represented by drawing a sketch, often called a "Venn" diagram.

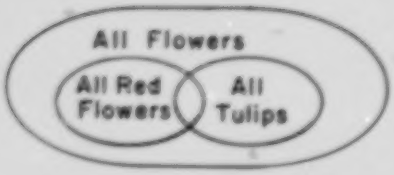


To write this relationship in mathematical language we use the symbol "$\subset$" which may be read "is a subset of" or "is contained in." You can now write: $B \subset C$.



The diagram at the right illustrates that the set of all red flowers is a subset of the set of all flowers. Let the set of all red flowers be called R and the set of all flowers be called F. The relationship of R and F can then be written as: $R \subset F$.

Note in the following Venn diagram that the set of all red flowers belongs to the set of all flowers, and that the set of all tulips also belongs to the set of all flowers:



Let the set of all tulips be called T. The above relationship may now be expressed as:

$$R \subset F, \text{ and}$$
$$T \subset F.$$

What can you say about the relationship of sets R and T ? You would certainly have to say that some tulips are red and are thus contained in set R. This is why the sets R and T are shown as overlapping ovals in the diagram. But you certainly cannot say that T R is true. Why not?

As another example let us find all the subsets of $B = \{1,3,5\}$. They would be: $\{1\}$, $\{3\}$, $\{5\}$, $\{1,3\}$, $\{1,5\}$, $\{3,5\}$, $\{1,3,5\}$, and the empty set, $\emptyset$. Any set is a subset of itself, and the empty set is considered to be a subset of every set. This may be a little clearer if you consider the set {Tom, Dick, Harry}, where we now think of the set of three boys whose names are Tom, Dick, and Harry, and not the set of three words--"Tom," "Dick," and "Harry." We now ask: "In how many ways could you ask none or some of the three boys to go to the ball game with you?" The answer is that you could ask any one of them, or any two of them, or all three of them, or none of them. Thus, the subsets are: {Tom}, {Dick}, {Harry}, {Tom, Dick}, {Tom, Harry}, {Dick, Harry}, {Tom, Dick, Harry}, and $\emptyset$.

We can state this concept of a subset in mathematical language as follows:

> If every element of a set S belongs to a set T, then S is said to be a subset of T. We say that S is contained in T; that is, $S \subset T$.

Also,

> S is a proper subset of T if $S \subset T$, $S \neq T$.

For example, the proper subsets of set $B = \{1,3,5\}$ would be all of the subsets of B except B itself; namely: $\emptyset$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1,3\}$, $\{1,5\}$, and $\{3,5\}$.

Sometimes the symbol $\subseteq$ is used to represent "is a subset of" and the symbol $\subset$ used only to represent "is a proper subset of."

Class Exercises

6. Draw Venn diagrams illustrating the following relationships:
   a. B is a proper subset of A.
   b. B and D are proper subsets of A, and B and D are disjoint.
   c. B and C are proper subsets of A, and B and C are not disjoint.

7. Given the sets $S = \{0,5,7,9\}$ and $T = \{0,2,4,6,8,10\}$.
   a. Find K, the set of all numbers belonging to both S and T. Is K a subset of S ? of T ? Draw a Venn diagram illustrating this.

b. Find M, the set of all numbers each of which belongs to S or to T or to both. (We never include the same number more than once in a set.) Is M a subset of S ? Is T a subset of M ? Is M finite?

c. Find R, the subset of M, which contains all the odd numbers in M. Of which others of our sets is this a subset?

8. The following table has been started:

	Set	Number of members	Subsets	Number of Subsets
a.	∅	0	∅	1
b.	{△}	1	∅, {△}	2
c.	{△, ○}	2	∅, {△}, {○}, {△,○}	4
d.	{△, ○, □}	3		
e.	{△,○,□,☆}			

How many different subsets can be formed from the members of the set in d ? From the members of the set in e ? Try to predict how many different subsets a set with eight members would have.

1.3 Intersection and Union of Sets

We often think of elements common to two sets. Suppose that in your class you asked all the boys who play in the school band to stand. Let this be the following set:

B = {Bill, Jim, Tom, Sam}.

Suppose these boys then sat down and you asked all the boys with red hair to stand. Let this be the following set:

R = {Sam, Tom, Carl}.

Finally, suppose you asked all the red-haired band members to stand. What would this set be? It would be the set:

{Tom, Sam}.

This set is called the intersection of set B and set R. The combining of two sets in this manner is an operation on these sets.

> The intersection (symbol, $\cap$) of two sets is the set of all elements common to each of the given sets.

Let us consider two other sets, G and H, defined as follows:

$$G = \{4,5,6,7,8,9\};$$
$$H = \{2,4,6,8,10\}.$$

From these two sets another set, K, may be formed whose members appear in both G and H:

$$K = \{4,6,8\}.$$

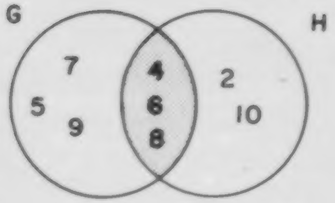
Set K consists of the members that sets G and H have in common and is therefore the intersection of the two sets. This may be written as:

$$\{4,5,6,7,8,9\} \cap \{2,4,6,8,10\} = \{4,6,8\}$$

or

$$G \cap H = K.$$

A diagram may also be used to illustrate this idea:



$G \cap H = \{4, 6, 8\}$

The shaded region indicates the intersection of the two sets.

Now consider set $R = \{1,2,3,4,5\}$ and set $S = \{6,7,8,9\}$. Sets R and S have no members in common (i.e., they are disjoint sets). Therefore, the intersection of the two sets is the empty set and we write $R \cap S = \emptyset$. Draw a Venn diagram illustrating this case.

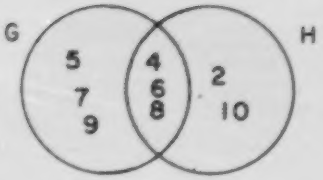
Another operation on sets is the combining of two sets in such a way that each of the members of the new set is in at least one of the two given sets. Recall again the members of the band and the red-haired boys. If all the boys who were either in the band or who had red hair were asked to stand, we would have the set:

{Bill, Carl, Jim, Tom, Sam}.

This is called the union of these two sets.

> The union (symbol, $\cup$) of two sets is the set of all elements that are in at least one of the given sets.

As another example, consider again set $G = \{4,5,6,7,8,9\}$ and set $H = \{2,4,6,8,10\}$. The union of set G and set H (written $G \cup H$) would be $\{2,4,5,6,7,8,9,10\}$ which we shall designate as set L. Therefore, $G \cup H = L$. A diagram such as the following may be drawn to illustrate this idea:



$G \cup H = L$

The shaded region shows the union of the two sets. (Remember, there is only one number 4, one 6 and one 8, therefore, 4, 6, and 8 are included only once in the union.)

Recall again set $R = \{1,2,3,4,5\}$ and set $S = \{6,7,8,9\}$. Then

$$R \cup S = \{1,2,3,4,5,6,7,8,9\}.$$

Can you illustrate this with a Venn diagram?

We would like to interject a note here again that much of this chapter is being presented as background information for teachers, and that most textbooks for students probably integrate these concepts as they are needed to develop some mathematical idea. However, some parts of this material may be presented to a class as a little side trip. Most youngsters enjoy this as something different and fairly easy to grasp.

As these ideas are presented many visual aids may be used. Sets of objects, plastic containers, and the use of overhead projectuals adapt themselves readily to this area. Different colored sheets of acetate cut in various shapes and placed on the stage of the overhead projector depict clearly the intersection and union operations. The student needs to be led to discover some mathematics for himself, and this topic is one in which this may be done quite effectively.

Class Exercises

9. Given: $A = \{b,d,e,f\}$
$B = \{a,b,c,d,e,f\}$
$C = \{a,b,c,e\}$
$D = \{a,c\}$
$E = \{d,f\}$
$F = \{b,e\}$

Which of the following sentences are true?

(a) $A \cup C = B$
(b) $B \cup C = B$
(c) $A \cup B = C$
(d) $B \subset C$
(e) $B \cap C = C$
(f) $F \subset F$
(g) $E \cup F = A$
(h) $A \cap C = F$
(i) $D \subset C$
(j) $(E \cup F) \subset (E \cap F)$
(k) $(D \cap E) \subset (D \cup E)$
(l) $D \cap E = F$

10. Let W = the set of all whole numbers.
E = the set of all even numbers.
O = the set of all odd numbers.

Describe each of the following sets:

(a) $E \cap O$
(b) $O \cup E$
(c) $W \cup E$
(d) $(E \cap O) \cup W$
(e) $(O \cap W) \cap E$
(f) $W \cup (E \cup O)$

1. Sentences, the Number Line, and Truth Sets

We need to pause for a moment in our discussion of sets and set language to talk about language in general, then see one of the applications for the use of sets. The teaching of mathematics not only must give the student a glimpse of the structure of the subject but must also treat the language with great care. The difference between words like "and" and "or," "if" and "only if," and "not" and "none," can mean the difference between understanding and misunderstanding.

Language also involves choice of descriptive words. Unlike the chemist, who uses long compound words to describe his materials, the mathematician often selects common words to describe uncommon concepts. The teacher should beware of dictionary meanings for words such as rational, real, imaginary, group, field, limit, term, factor, range. When these words are used as mathematical terms, they do not have the meanings commonly ascribed to them.

On the other hand, teachers of junior high school youngsters need to be careful of what is expected from their students in the way of verbal responses. Certainly textbooks and teachers need to be precise in their language, but perhaps the mind of a seventh grader has not sufficiently matured to enable him to make statements in as precise mathematical language as we would wish. This is one of the things we are trying to train him to do! We must keep in mind the following question: "Are we communicating with our students, and are they communicating with each other?"

Mathematicians now make use of the structure of English sentences to communicate mathematical concepts. For example, the English sentence, "He was the first president of the United States" is neither true nor false until we give a replacement for "He." This sentence is called an open sentence. It may be true: "George Washington was the first president of the United States," or false: "Abraham Lincoln was the first president of the United States." In fact, "☐ was the first president of the United States" may be a test question requiring the name of the man for which it would be a true sentence. Open number sentences are the basis of a great deal of work in mathematics. For example, consider the following mathematical sentences:

(a) $3 + 5 = 8$
(b) $9 - 2 = 6$.
(c) $5 + \square = 7$
(d) $2n + 6 = 9$

Sentence (a) is a true mathematical sentence, (b) is a false sentence, and (c) and (d) are open mathematical sentences, being neither true nor false.

All sentences require verbs. Some of the most common ones in mathematics are listed in the table below.

Symbol	Verb	Example
=	"is equal to"	$3 + 4 = 7$
≠	"is not equal to"	$5 - 2 \neq 4$
>	"is more than"	$7 - 3 > 1$
<	"is less than".	$5 < 10$
≥	"is more than or equal to"	$9 \geq$ any one-digit number
≤	"is less than or equal to"	$0 \leq$ any whole number

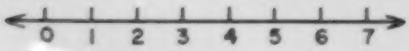
None of the examples listed above is an open sentence. They all make true statements about specific numbers which are described or represented by a single numeral such as 7 or by a *mathematical* or number phrase such as $3 + 4$. If we want to write an open number sentence, we need to use an *open number phrase* such as $\square + 7$ or $17 - \square$, where the symbol $\square$ is used to help you remember that the empty space is to be filled by some numeral from a given set. Because symbols like $\square$ are awkward to type or write, we frequently use letters such as n, s, x, or y for the same purpose. Thus, a simple open number phrase may be written as $n + 7$ instead of $\square + 7$, and an open number sentence as $n + 7 = 10$. What whole number or numbers will now make this open sentence a true statement? In this case the answer is easily obtained by trial: $3 + 7 = 10$, while $0 + 7 \neq 10$, $1 + 7 \neq 10$, $2 + 7 \neq 10$, $4 + 7 \neq 10$. We see that 3 is the only number which does the trick. It is the only replacement for n that will make the sentence true.

What whole number or numbers will make the open sentence $x < 5$ a true statement? Again, by trial we find that $0 < 5$, $1 < 5$, $2 < 5$, $3 < 5$, and $4 < 5$, are true statements while $5 < 5$, $6 < 5$, $7 < 5$, and so on, are false statements. Thus, we see that from the set of whole numbers $\{0,1,2,3,4,5,6,\ldots\}$ only the members of the set $\{0,1,2,3,4\}$ make the statement true.

What about the open sentence $n + 7 < 11$? We can translate the sentence into words by saying "the sum of a certain number and 7 is less than 11." The whole numbers which make this a true statement of inequality are the members of the set $\{0,1,2,3\}$. This set of whole numbers is called the *truth set* or *solution set* of the open sentence $n + 7 < 11$. Sentences with the verb "$=$" are called *equations*, whereas sentences with any of the other verbs listed above are called *inequalities*.

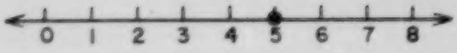
Another very useful device in our study of number sentences is to establish a one-to-one correspondence between the set of whole numbers, $W = \{0,1,2,3,4,\ldots\}$, and a set of certain points on a line. In a later chapter we will associate all the points on a line with the set R of all *real* numbers. We simply draw a picture of a line with arrows on both ends.

Starting at an arbitrary point that we label 0, we mark off equally-spaced intervals that are labeled with the set of whole numbers in order:



We call the number corresponding to a point the coordinate of that point. The order of the whole numbers shows up clearly by the position of the marks; $5 > 3$ indicates that the coordinate of 5 is to the right of the coordinate of 3.

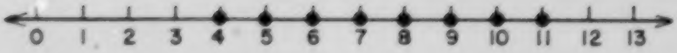
Now a picture of a solution set using the number line can be drawn. Consider the open sentence $x + 3 = 8$. This open sentence has only the one solution, 5. (A solution is an element of the solution set.) Thus, the solution set is $\{5\}$. On the number line this solution can be represented as shown below:



$x + 3 = 8$

The solution, 5, is indicated by a solid dot.

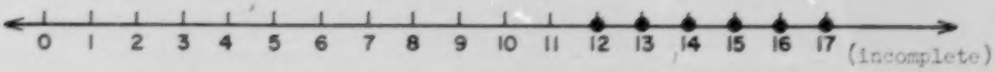
In the examples that follow we shall restrict our discussion to whole numbers only. The solution set of the inequality $n - 4 \leq 7$ can then be represented thus:



$n - 4 \leq 7$

Notice that if n represented the number 3, 2, 1, or 0, then the open number phrase $n - 4$ would represent a negative integer. Since we have restricted our discussion to the whole numbers only, these numbers are not considered as part of the solution set.

Note that on the number line we indicate the solution set by heavy solid dots. The solution set of $n - 4 > 7$ cannot be completely represented this way because it consists of all whole numbers greater than 11. However, we can represent it by heavy dots up to the arrow and the word "incomplete" to show that all the whole numbers represented by points still further to the right are also members of the solution set.



$n - 4 > 7$

Other notations are sometimes used to illustrate this same type of solution set on a number line. Pictures of solution sets on the number lines are called the graphs of the solution sets or truth sets of the respective mathematical sentences.

Class Exercises

Give the solution set of each of the following sentences, using the set of whole numbers. Then represent each solution set on a number line.

11. $x + 7 = 9$
12. $5y < 7$
13. $n + 6 < 1$
14. $3 > 4 - p$
15. $x + 4 \leq 6$
16. $x + 4 > 6$
17. $7z + 5 \leq 30$
18. $3x - 6 < 5$

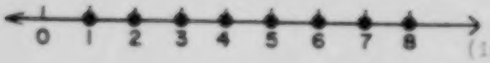
1.5 Compound Number Sentences

The graphing of simple inequalities, as illustrated in the last section, is also a convenient way to find and picture the solution set of compound number sentences. Such sentences are formed by combinations of two or more simple sentences and the connectives "and" and "or." Examples of such compound sentences are given below:

(a) $x \geq 1$ and $x < 5$
(b) $x \leq 1$ or $x > 5$
(c) ($x > 5$ and $x \leq 7$) or $x < 3$.

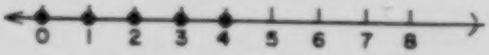
Recalling the definition of intersection and union of sets will help us in finding the solution sets for such compound sentences. Example (a) will be true when both simple sentences are true. This means that the solution set we are seeking is the intersection of the solution sets of the two simple sentences considered separately. The two solution sets, again using only the whole numbers, are readily found and graphed.

$x \geq 1$



(Incomplete)

$x < 5$

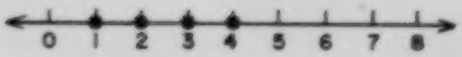


The solution set of $x \geq 1$ is $\{1,2,3,4,\ldots\}$ while the solution set of $x < 5$ is $\{0,1,2,3,4\}$. We may show the solution set of the compound sentence by set notation

$$\{1,2,3,4,\ldots\} \cap \{0,1,2,3,4\} = \{1,2,3,4\}$$

or by a graph:

$x \geq 1$ and $x < 5$



This method of solution by graphing is very useful when the present restriction of whole numbers only is removed. More complicated inequalities such as

$$x^2 - x - 6 \geq 0$$

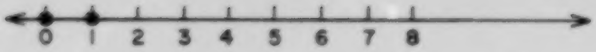
are easily solved by techniques very much like these.

Compound number sentences involving "or" may be solved in a similar way. This time we recall the definition of union of sets and see that the solution of the sentence

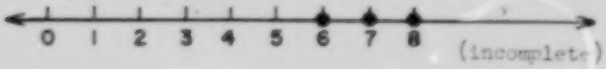
$$x \leq 1 \text{ or } x > 5$$

is the union of the solution sets of $x \leq 1$ and $x > 5$. This is the case since the statement, $x \leq 1$ or $x > 5$, is true when at least one of the two simple sentences is true. Using the graphical method gives us the following:

$x \leq 1$



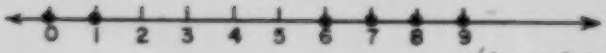
$x > 5$



(incomplete)

The solution set of $x \leq 1$ is $\{0,1\}$ and the solution set of $x > 5$ is $\{6,7,8,\ldots\}$. Their union is $\{0,1,6,7,8\ldots\}$. Thus, the graph of the solution set to the compound sentence is

$x \leq 1$ or $x > 5$



(incomplete)

When confronted with a compound sentence formed from three or more simple sentences the technique is much the same. As an example, the solution set for the sentence

$$(x > 5 \text{ and } x \leq 7) \text{ or } x < 3$$

may be found by first finding the solution set for the statement within the parentheses and then combining that with the solution set for $x < 3$. The solution set for the compound sentence within the parentheses is $\{6,7\}$. This, when combined with the solution set of $x < 3$, $\{0,1,2\}$, gives $\{0,1,2,6,7\}$.

Step 1:

$x > 5$

Step 2:

$x \leq 7$

Step 3:

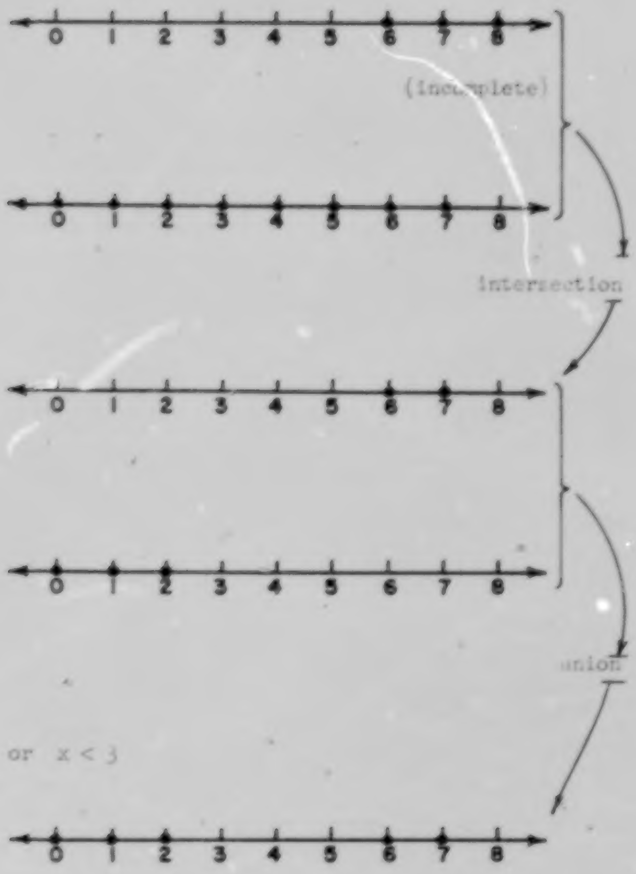
$x > 5$ and $x \leq 7$

Step 4:

$x < 3$

Solution:

$(x > 5$ and $x \leq 7)$ or $x < 3$


(incomplete)
Intersection
union
0 1 2 3 4 5 6 7 8

Class Exercises

Find the solution set for each of the following sentences using whole numbers. Show each solution on a number line.

19. $x > 5$ and $x < 7$
20. $x > 5$ or $x < 7$
21. $x < 5$ and $x = 7$
22. $x \leq 5$ or $x \geq 7$
23. $x \geq 5$ and $x \neq 7$
24. $(x < 3$ or $x \geq 5)$ and $x \leq 6$
25. $(x < 3$ and $x \geq 5)$ or $x = 5$

1.6 Conclusion

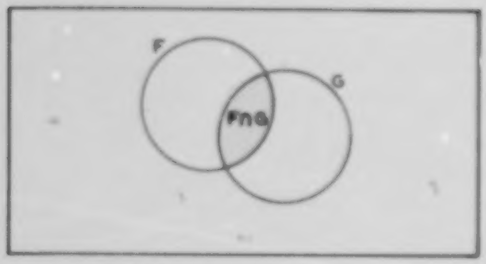
As we stated earlier in this chapter, the language and the properties of sets will be used throughout this book whenever these enable us to expedite our presentation of mathematical ideas about numbers, operations, and geometry.

Let us look again at one of the counting problems from the introduction and analyze it using set language and Venn diagrams. Then we will leave one for you as an exercise. Sometimes it is more important that we find ten ways to do one problem than to find one way to do ten problems!

We will illustrate the first counting problem about the class of 35 pupils, all taking a foreign language. Twenty-one are enrolled in French and 17 in German. How many students are enrolled in both French and German?

Call the pupils taking French set F, and the pupils taking German set G. Then the number of pupils enrolled in both French and German would be the number in the intersection of these two sets: $F \cap G$.

We may draw a diagram of this, the shaded portion being the intersection:



Now, the number of members in set F is 21, and the number of members in set G is 17. Let us denote this by $n(F) = 21$, and $n(G) = 17$.

Now,

$$n(F \cup G) = 35$$

since all 35 pupils are taking at least one foreign language. Also,

$$n(F) + n(G) - n(F \cap G) = 35.$$

Do you agree?

Substituting in the above equation we obtain:

$$21 + 17 - n(F \cap G) = 35;$$

and

$$n(F \cap G) = 3.$$

Therefore, the number in the intersection of the sets, or the number of pupils taking both languages, is 3.

From the diagram it is how a simple matter to find the number of pupils taking only French and the number of pupils taking only German. How many are there in each of these sets?

This is a fairly simple problem and the explanation using set notation may seem longer than the explanation given for the same problem in the Introduction. However, with a little experience you will find that more complicated problems of this nature lend themselves easily to illustrations with Venn diagrams and to solutions with set language.

Class Exercises

26. Illustrate the following problem with a Venn diagram and then solve it using set notation.

> In a housing development of 1024 houses, 795 houses have a tree in the front yard and 844 houses have a tree in the back yard. If it is known that every house has at least one tree, how many houses have a tree in both the front and back yard?

Chapter Exercises

1. Write all possible subsets of the set: $\{4,5,6\}$.
2. Draw a Venn diagram to illustrate the following:
   (a) The set whose elements are the Hudson and Ohio Rivers is contained in the set of all rivers in the United States.
   (b) The set whose elements are all tigers, lions, and baboons is contained in the set of all animals.
   (c) The set of numbers 16, 36, and 40 is contained in the set of all counting numbers which are multiples of 4.
3. Given three sets A, B, and C. If $B \subset A$ and $C \subset B$, is $C \subset A$? Illustrate your answer with a Venn diagram.

4. Given the three sets: $A = \{\text{boy}, \text{girl}, \text{chair}\}$, $B = \{\text{girl}, \text{chair}, \text{dog}\}$, and $C = \{\text{chair}, \text{dog}, \text{cat}\}$.
   - (a) Find $A \cap B$.
   - (b) Show that $A \cap C = C \cap A$.
   - (c) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   - (d) Show that $A \cap (B \cap C) = C \cap (A \cap B)$.

5. (a) Let $\emptyset$ represent the null set, and H any other set. Is the statement $\emptyset \cup H = H \cup \emptyset$ true? Explain your answer.
   - (b) Is $\emptyset \cap H = H$? Explain your answer.

6. Let A be the set of even counting numbers; B the set of odd counting numbers; and C the set of all counting numbers.
   - (a) Is $A \cup B = C$? Why?
   - (b) Is $A \subset C$? Why?
   - (c) Is $A \subset B$? Why?
   - (d) Is $A \cup B = B \cup A$? Why?
   - (e) Is $B \subset A$? Why?
   - (f) Draw a Venn Diagram to illustrate $B \subset C$.
   - (g) Is $A = B$? Why?

7. (a) Let E be the set of even counting numbers: $\{2,4,6,8,\ldots\}$. Describe a set F so that $E \cup F = C$, when C is the set of all counting numbers.
   - (b) What is $E \cap F$?
   - (c) Could set F be some other set than what you named in (a)?

8. Given two sets A and B:
   - (a) If $A \subset B$, is it true that $A \cup B = B$? Explain your answer.
   - (b) If $A \subset B$, is it true that $A \cap B = A$? Explain your answer.

9. Suppose you buy a carton of a dozen eggs. Is it necessary to count the eggs in order to tell whether or not you have a dozen? Why?

10. Draw a Venn diagram to illustrate the problem from the introduction about the 225 students taking foreign languages.

11. Sometimes a many-to-one correspondence between two sets may be defined. If $S = \{1,2,3,\ldots,31\}$ and $T = \{\text{Sun.}, \text{Mon.}, \text{Tues.}, \ldots, \text{Sat.}\}$, with the correspondence given by the calendar for July of any one year, show how a many-to-one correspondence may be established.

12. Find the solution set for each of the following sentences using whole numbers. Show each solution on a number line.

(a) $x > 2$ and $x < 8$.
(b) $x \geq 9$ and $x < 10$.
(c) $x \geq 4$ or $x < 4$.
(d) $x < 9$ or $x > 12$.
(e) $(x > 5$ and $x < 12)$ or $x \leq 4$

13. Given set W of the whole numbers and set O of the odd whole numbers, show with a diagram how a one-to-one correspondence may be set up between these sets. What other observations can you make about the matching of these two infinite sets?

Answers to Class Exercises

1. a. Yes
   b. No
   c. Yes
   d. Yes
   e. Yes

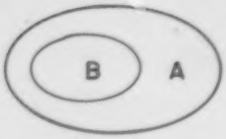
2. Only two examples are given here, but certainly there are others, and answers will vary.
   a. $\{1,3,5,7,9,11\}$, or the set of odd whole numbers less than 12.
   b. The set of counting numbers which are multiples of 10 and also less than 101, or the set of multiples of 10 between 0 and 110.
   c. $\{51,52,53,54,\ldots\}$, or the set of whole numbers beginning with 51.
   d. $\emptyset$, or the set of boys in your class over 12 feet tall.

3. a. True. 3 is a member of $\{2,3,4,5\}$.
   b. False. $\{0\}$ has the number zero in it whereas $\emptyset$ has no members.
   c. False. $\{\emptyset\}$ has a member, namely, $\emptyset$. $\emptyset$ has no members.
   d. False. $\{5,6,7,8,\ldots\}$ is an infinite set containing 17.
   e. False. They both include exactly the same members and are therefore equal. The order in which the members are listed is immaterial.
   f. False. The set contains all multiples of 4 less than 100 and thus includes 32.

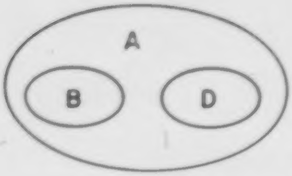
4. a. Infinite.
   b. Finite. It is the empty set.
   c. Finite.
   d. Finite. It is the empty set.
   e. Finite.

5. The elements of set M are 3, 5, 7, 11, 13, 17, 19 and 23. Did you say the odd numbers from 3 through 23 ? Set M was conceived as the set of the first eight odd prime numbers. Moral: One must be very careful when using this notation.

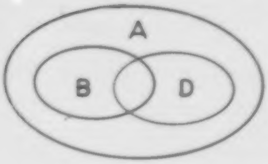
6. a.



b.

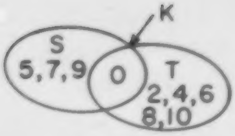


c.



(There are other possibilities)

7. a. $K = \{0\}$. Yes. Yes.
   b. $M = \{0, 2, 4, 5, 6, 7, 8, 9, 10\}$. No. Yes. Yes.
   c. $R = \{5, 7, 9\}$. R is a subset of S and M.

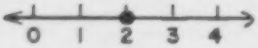

K
S
5,7,9
0
T
2,4,6
8,10

8. Eight subsets. Sixteen subsets. 256 subsets. (This takes the form of 2^n where n is the number of members of a set.)

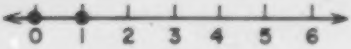
9. a.	True	g.	True
b.	True	h.	True
c.	False	i.	True
d.	False	j.	False
e.	True	k.	True
f.	True	l.	False

10. a.	$\emptyset$	d.	W
b.	W	e.	$\emptyset$
c.	W	f.	W

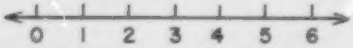
11. $\{2\}$



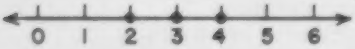
12. $\{0,1\}$



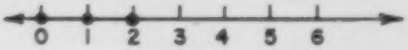
13. $\emptyset$



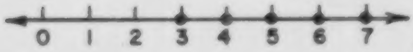
14. $\{2,3,4\}$



15. {0,1,2}



16. {3,4,5,6,...}



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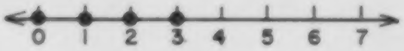
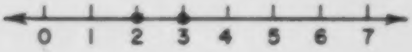
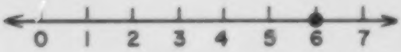
17. {0,1,2,3}



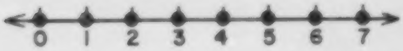
18. {2,3}



19. {6}



20. {0,1,2,3,...}



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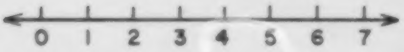
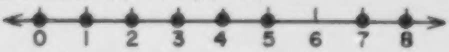
21. ∅



22. The set of all whole numbers except 6.



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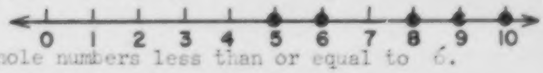
23. {5,6,8,9,10,11,12,13,14,15,...}


0 1 2 3 4 5 6 7 8 9 10

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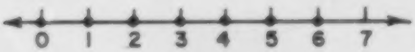
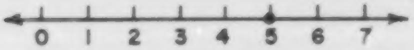
24. The set of all whole numbers less than or equal to 6.



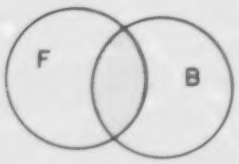
25. {5}



26. Let H = the set of houses in the development, then n(H) = 1024.

27. Let F = the set of houses with a tree in the front yard, then n(F) = 795.

Let B = the set of houses with a tree in the back yard, then n(B) = 844.



$$\begin{aligned} n(F \cup B) &= 1024 \\ n(F) + n(B) - n(F \cap B) &= 1024 \\ 795 + 844 - n(F \cap B) &= 1024 \\ n(F \cap B) &= 615 \end{aligned}$$

Chapter 2

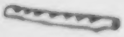
NUMERATION

Introduction

Systems of numeration are invented by men to meet their needs to express number ideas. As civilization has increased these needs, older numeration systems have either expanded or given way to improved systems. Studying the history of the earlier systems provides background for teachers to teach the structure of the decimal system of numeration, deepens pupil's understanding of the principles of numeration, and provides a vehicle for review that is not repetitious.

Through experiences in reading and writing names for large numbers, in using exponents to write new names for numbers, and in using the expanded form for representing numbers, pupils learn that the decimal system may become a powerful vehicle for them. Some familiarity with number bases other than ten makes possible a comparison of these systems with the decimal system, thus reinforcing the understanding of the decimal system. Teachers and pupils who experience "building" a system to a base other than ten no longer take for granted the decimal system of numeration. One additional point should be emphasized here. While the teacher should fully understand the material of this chapter, he must guard against spending too much time in teaching this material to his junior high school students. The understanding of the concept of place value and the relationship of non-ten bases to the decimal system are important. But the students should not be required to memorize the tables of operations or to master completely the skills of computing in different bases. The topic is useful but is not essential for a modern mathematics program.

2.1 Early Numeration Systems

In primitive times men were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows." Their language indicates that they had not learned abstract words for number ideas. Primitive peoples learned to use numbers to keep records. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in sticks to represent the number of objects counted. A boy counting sheep would have pebbles, or he might make notches in a stick, as . Each pebble or mark in the stick would represent a single sheep in a one-to-one correspondence between pebbles and sheep. The same kind of record is made when votes in a class election are tallied, as 𝍸 // .

As centuries passed, early people used sounds, or names, for numbers. Today standard sets of names for numbers are used. A rancher counting sheep compares a single sheep with the name "one," and 2 sheep with the name "two" and so on. Man now has symbols (1, 2, 3, ...) and words (eins, zwei, drei,...) which may be used to represent numbers. Word names for numbers vary with the language spoken. For example,

Hindu-Arabic Numerals:	1	2	3
English	one	two	three
German	eins	zwei	drei
Spanish	uno	dos	tres
Latin	unus	duo	tres

A numeration system is a way of naming numbers. It has some basic numerals, and it has ways for making other numerals from them. Different systems of numeration have evolved as their need has arisen.

It is essential that the terms number and numeral be clearly understood. The words are not synonymous. A number is a concept, an idea, an abstraction. A numeral is a symbol, a name for a number. A numeration system is a numeral system, not a number system. It is a system for naming numbers.

Of the ancient systems of numeration, perhaps the Egyptian, Babylonian, Early Roman, Chinese, and Mayan are the best known. A study of Egyptian numerals is given here; it is suggested that other early systems may be introduced by using references, library assignments, class reports, and group projects. Activities of this nature may be correlated with social studies by using the historical period as framework for a system of numeration.

Egyptian System of Numeration

One of the earliest systems of writing numerals for which there is some record is the Egyptian system. Their hieroglyphic, or picture, numerals have been traced as far back as 3300 B.C. Thus, more than 5000 years ago, Egyptians had developed a system with which they could express numbers up to the millions.

The basic Egyptian symbols are shown below:

Our Numeral	Egyptian Symbol	Object Represented
1	𓏺	stroke or vertical staff
10	𓎆	heel bone
100	𓍢	coiled rope or scroll
1000	𓆼	lotus flower
10,000	𓂭	pointing or bent finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

These symbols were carved on wood or stone. The Egyptian system was an improvement over earlier primitive systems because it incorporated the following ideas:

1. A single symbol could be used to represent the number of objects in a collection. For example, the heel bone represented the number ten.
2. The basic symbols could be repeated within a given numeral. For example, the group of symbols 𓍢𓍢𓍢 meant 100 + 100 + 100 or 300.
3. This system was based on groups of ten. Ten strokes are equivalent to a heelbone, ten heelbones are equivalent to a scroll, and so on.

The following table shows how the Egyptians represented certain numerals.

Our Numeral	4	11	23	20,200	1959
Egyptian Numeral	𓏺𓏺𓏺𓏺	𓎆𓏺	𓎆𓎆𓏺𓏺𓏺	𓂭𓂭 𓍢𓍢	𓆼 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢 𓎆𓎆𓎆𓎆𓎆 𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺

Note that each basic Egyptian symbol means the same thing regardless of where it is placed. For example, 𓎆𓏺𓏺 , 𓏺𓏺𓎆 , and 𓏺𓎆𓏺 all represent the same number, twelve. This is a significant difference from our numeration system where position plays an important role. In our numeration system, the numerals "12" and "21" do not name the same number.

Other Ancient Systems of Numeration

It is likely that most students at this level are familiar with the early Roman system of numeration. If this be the case, then it would be helpful to compare the Roman system to both the Egyptian and the present decimal system of numeration. While based on grouping by tens, Roman numeration also incorporates a modified grouping by fives as illustrated in the table below.

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

In early times the Romans repeated symbols in their numerals the same way that the Egyptians had done many years before. Later, the Romans made use of subtraction to shorten some numerals. Recall that the values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral. When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger. The better student may be interested in exploring on his own some of the other ancient systems of numeration.

Each of the early numeration systems was an improvement over matching objects with notches or pebbles. While it is fairly easy to represent a number in any of the early systems, it is difficult to use the numerals for computing such as in addition and multiplication. It is not as important that students learn to manipulate these numerals as that they learn enough about the systems to compare them with the decimal system of numeration.

Class Exercises

1. How did the Egyptians represent the numbers

   (a) 100; 1000; 10 ? (b) 103; 501 ?

2. Write several arrangements of the Egyptian numeral for the number 1,234.

3. How would you add

   (a) ∩∩||| and ∩∩∩|||| ?

   (b) ∩∩∩∩∩∩∩ ||||| and ∩∩∩∩ |||||||| ?

4. Can you devise a plan for multiplication using Egyptian numerals? Try it with ∩∩||| times ||||| .

2.2 Expanded Notation and Exponents

There are many instances in mathematics in which we use a certain number more than once as a factor. Examples are found in the computation of the area of a square, $A = s \times s$; in the volume of a cube, $V = e \times e \times e$; and in the volume of a sphere, $V = \frac{4}{3} \times \pi \times r \times r \times r$.

Another illustration of the use of a number several times as a factor is found in our decimal place value system of numeration. The value represented by a digit in a decimal numeral depends upon the position of the digit in the numeral. Note the different values represented by the two digits "4" in the following example.

$$\begin{aligned} 1484 &= (1 \times 1000) + (4 \times 100) + (8 \times 10) + (4 \times 1) \\ &= (1 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (8 \times 10) + (4 \times 1) \end{aligned}$$

Teachers are already familiar with the use of place value in the decimal system as shown in the table on the next page. However, some may not be familiar with writing powers of ten in exponential form. For this reason, an explanation of exponents is given in this section.

Frequently, place values for the decimal system are written more briefly by using the exponential form. In general, the exponent shows how many times the base is used as a factor in a product. Values of the places are read as follows:

1,000,000	10^6	"Ten to the sixth power"
100,000	10^5	"Ten to the fifth power"
10,000	10^4	"Ten to the fourth power"
1,000	10^3	"Ten to the third power"
100	10^2	"Ten to the second power"
10	10^1	"Ten to the first power"
1	10^0	"One"

Numeral	Exponential Form	Place Value	Group Name
1,000,000,000	10^9	10 × 10 × 10 × 10 × 10 × 10 × 10 × 10 × 10	Billion
100,000,000	10^8	10 × 10 × 10 × 10 × 10 × 10 × 10 × 10	Hundred Million
10,000,000	10^7	10 × 10 × 10 × 10 × 10 × 10 × 10	Ten Million
1,000,000	10^6	10 × 10 × 10 × 10 × 10 × 10	Million
100,000	10^5	10 × 10 × 10 × 10 × 10	Hundred Thousand
10,000	10^4	10 × 10 × 10 × 10	Ten Thousand
1,000	10^3	10 × 10 × 10	Thousand
100	10^2	10 × 10	Hundred
10	10^1	10	Ten
1	10^0	1	One

Decimal System Place Value Table

All the numbers represented above are called powers of ten. In 10^6 the 6 is the exponent and the 10 is the base. In 10^1 the 1 is the exponent. Since 10^1 equals 10, the exponent 1 is frequently omitted. However, all other exponents must be written when expressing powers of ten in exponential form.

In 10^0 the 0 is the exponent. Notice that 10^0 has been defined here as being equal to one. In general, we agree to define $a^0 = 1$ for any number a except 0. The convenience of this definition will become apparent later in this section.

The use of exponents enables us to shorten the expanded form of decimal numerals as illustrated below.

$$\begin{aligned} 2603 &= (2 \times 1000) + (6 \times 100) + (0 \times 10) + (3 \times 1) \\ &= (2 \times 10 \times 10 \times 10) + (6 \times 10 \times 10) + (0 \times 10) + (3 \times 1) \\ &= (2 \times 10^3) + (6 \times 10^2) + (0 \times 10^1) + (3 \times 10^0) \end{aligned}$$

The various forms of representing the number 2603 illustrate the use of expanded notation. Writing a numeral in expanded notation explains the meaning of each digit in the numeral. The form using exponents is sometimes simplified, replacing 10^0 by 1 as shown here.

$$\begin{aligned} 32{,}750 &= (3 \times 10^4) + (2 \times 10^3) + (7 \times 10^2) + (5 \times 10^1) + (0 \times 1) \\ &= \quad 30{,}000 \quad + \quad 2000 \quad + \quad 700 \quad + \quad 50 \quad + \quad 0 \end{aligned}$$

Not only are exponents useful in simplifying the writing products; they greatly simplify certain computations. Some examples of computation will indicate the value of using exponents. In each case note the relationship between the exponents of the factors and the exponent of the result.

With 10 as the base:

$$10^1 \times 10^4 = 10 \times (10 \times 10 \times 10 \times 10) = 100{,}000 = 10^5$$

$$10^2 \times 10^2 = (10 \times 10) \times (10 \times 10) = 10{,}000 = 10^4$$

$$10^0 \times 10^3 = 1 \times (10 \times 10 \times 10) = 1{,}000 = 10^3$$

With 2 as the base:

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 32 = 2^5$$

$$2^2 \times 2^1 = (2 \times 2) \times 2 = 8 = 2^3$$

With 5 as the base:

$$5^2 \times 5^3 = (5 \times 5) \times (5 \times 5 \times 5) = 3125 = 5^5$$

$$5^1 \times 5^3 = 5 \times (5 \times 5 \times 5) = 625 = 5^4$$

Likewise, with a as the base:

$$a^2 \times a^2 = (a \times a) \times (a \times a) = a^4$$

$$a^1 \times a^4 = a \times (a \times a \times a \times a) = a^5$$

$$a^3 \times a^0 = (a \times a \times a) \times 1 = a^3$$

Each of these examples illustrates the property that the product of two powers of a given base can be expressed in exponential form by adding the original exponents. In symbols, this law may be stated as

$$a^m \times a^n = a^{m+n} .$$

Thus, for example, $10^5 \times 10^6 = 10^{5+6} = 10^{11}$. Notice that this law holds when one or both of the original exponents is zero. For example, $10^3 \times 10^0 = 10^{3+0} = 10^3$. Hence, our agreement to define $a^0 = 1$ when $a \neq 0$, makes this law more general.

When it is necessary to use large numbers, working with exponents is especially convenient. For example, the number 7,000 may be expressed as $7 \times 1{,}000 = 7 \times 10^3$; the number $800{,}000 = 8 \times 100{,}000 = 8 \times 10^5$; and the number $63{,}000 = 63 \times 1000 = 63 \times 10^3$. The product $7{,}000 \times 800{,}000$ may be written as: $(7 \times 10^3) \times (8 \times 10^5) = (7 \times 8) \times (10^3 \times 10^5)$.

The earth's weight is approximately 13,000,000,000,000,000,000,000 pounds. Now this is a large number to read, to write, or to use in computation. It may be written as 13×10^{21}. Astronomers use ninety-three million miles as the mean distance from the earth to the sun. This number may be expressed by the numeral 93,000,000 , by the product expression $93 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, or by the exponential form 93×10^6. The exponential form of this number is convenient for most purposes.

Class Exercises

5. Write the following numbers in expanded notation using the exponential form:

   (a) four hundred thirty-six

   (b) five thousand, four hundred nine

   (c) thirty-three thousand, nine hundred eighty-seven

   (d) five million, two hundred fifty-six thousand, eight hundred ninety-eight.

6. Supply the missing parts in the table.

A Decimal Numeral	B Product Expression with Repeated Factors	C Exponential Form	D Powers of Ten
(a) 100	10 × 10	10^2	
(b) 10,000			Fourth
(c)	10 × 10 × 10 × 10 × 10		
(d)			Sixth
(e) 100,000,000			

7. Write in standard form the numeral indicated by:

   (a) (4×10^3)

   (b) $(3 \times 10^6) + (2 \times 10^4) + (5 \times 10^2)$

   (c) $(5 \times 10^5) + (4 \times 10^3) + (3 \times 1)$

   (d) $(6 \times 10^3) + (0 \times 10^2) + (0 \times 10^1) + (6 \times 1)$

8. The earth's weight was given as about 13,000,000,000,000,000,000,000 pounds. Express the weight of the earth in exponential form. A pound is approximately equal to 2.2 kilograms. What is the weight of the earth in kilograms?

9. Did you ever hear the name "googol" used for a number? Googol is the name given to a number written as "1" followed by one hundred zeros. Express this number as a power of 10.

2.3 Numeration in Other Bases

There are many familiar activities that utilize the concept of grouping of numbers other than by tens. Questions such as these, chosen from activities of daily living rather than from the context of a mathematics book, may act as a springboard for this section.

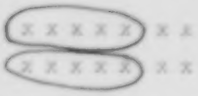
How many eggs are 2 dozen eggs?
How many nickels in 2 quarters?
How many persons sing in a double octet?
How many days are in 2 weeks?
How many persons compose 2 tables of bridge?
How many wheels are on 2 tricycles?
How many shoes are in 2 pairs?
How many trousers in 1 pair?

By investigating other number bases, we become more aware of how our decimal system works.

Base Five

In studying the decimal system of numeration, we grouped sets of objects by tens, and chose names for the place values that were based on powers of ten. The decimal system of numeration is a "base 10" system using ten symbols for building the system.

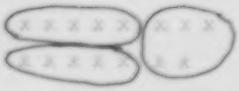
Let us look at a "base 5" system which groups sets of objects by fives. In this system, powers of 5 determine the place values, and five symbols, such as 0, 1, 2, 3, 4 are used for building numerals in the system. Suppose the set of 14 objects represented by x's in the figure below are grouped into sets of fives and ones as shown.



2 sets of fives and
4 sets of one

This grouping may be recorded as 2 fives and 4 ones, or more simply by the numeral 24_{five}. This numeral is read "two four, base five." It is necessary to use the written subscript "five" to designate the base five grouping.

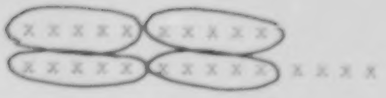
If one more x is added to the set shown above, we would have the following when grouping by fives:



3 sets of fives and
0 sets of one

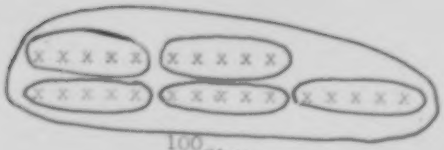
We can represent the number of x's in this set as 30_{five}. This is read as "three zero, base five," and represents the number fifteen.

All grouping in the base five system is by powers of five. Thus, if an x is added to the set shown at the left, we could group by fives to get the figure at the right.



44_{five}

(4 groups of fives and 4 ones)



100_{five}

(1 group of five × five and 0 fives and 0 ones)

In base five, we rewrite 5 ones as 1 five, 5 fives as 1 twenty-five, and so on. In the figure at the right above, if five is considered a group, then 5 fives may be associated with a group of groups.

Place values in base five numeration are powers of five. Notice how the powers of five are used in expressing 1231_{five} in expanded form.

$$\begin{aligned}1231_{five} &= (1 \times 125) + (2 \times 25) + (3 \times 5) + (1 \times 1)\\ &= (1 \times 5 \times 5 \times 5) + (2 \times 5 \times 5) + (3 \times 5) + (1 \times 1)\\ &= (1 \times 5^3) + (2 \times 5^2) + (3 \times 5^1) + (1 \times 5^0)\end{aligned}$$

Note that in this example the expanded notations use base ten symbols. One might write 1231_{five} as

$$1_{five} \times (10_{five})^3 + 2_{five} \times (10_{five})^2 + 3_{five} \times (10_{five})^1 + 1_{five} \times (1_{five})$$

in the expanded notation. For psychological reasons we use the notation involving base ten numerals; most of us think quickly and with more feeling in base ten.

But what does the numeral 1231 mean when expressed in other bases?

$$\begin{aligned}1231_{ten} &= (1 \times 10^3) + (2 \times 10^2) + (3 \times 10^1) + (1 \times 10^0)\\ 1231_{seven} &= (1 \times 7^3) + (2 \times 7^2) + (3 \times 7^1) + (1 \times 7^0)\end{aligned}$$

Normally, when a numeral is written in base ten the subscript word "ten" is omitted. In the remainder of this chapter, if no base is indicated with a numeral, it can be assumed to be in base ten.

Teachers are encouraged to limit the time devoted to the study of number bases other than ten. It is important for pupils to strengthen their background by comparing the structure of such systems with that of the decimal system. Place value and other concepts of structure are learned from the study.

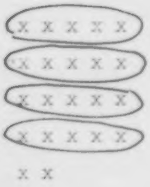
Quarters, nickels, and pennies may be used to illustrate examples in base five using no more than three places (digits). Since 5 pennies equal 1 nickel, and 5 nickels equal 1 quarter, grouping of these coins lends itself to grouping in base five.

Class Exercises

10. Look at the example in the diagram.

(a) How many sets of five x's are shown? How many single x's remain?

(b) Express the number of x's as a base five numeral. Then read the base five numeral.



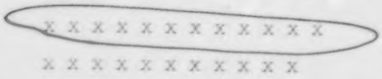
11. Complete the chart:

	Base Five Numeration Sets	Base Five Numeral
x x x x x x x x x x x x x x x x x x x x x x x x	____fives____ones	________
x x x x x x x x x x x x x x x x x	____fives____ones	________
x x x x x x x x x x x x x x x	____fives____ones	________

Base Twelve

In the base ten system of numeration the place values are powers of ten and ten digits are needed. In the base five system, the place values are powers of five and five digits are needed.

Consider next a base twelve system of numeration. It follows that the place values in this system would be powers of twelve and that twelve different digits would be needed. This means that in addition to the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 we must assign two extra digits, say T and E, to represent ten and eleven. To represent twenty-three, the number of x's in the diagram below, as a base twelve numeral, we would need to group by twelves.



Since we get one group of twelve and eleven ones, we would write $1E_{twelve}$. Some other base twelve numerals are listed here. See if you can verify the value of each.

$$\text{sixty-two} = 52_{twelve} = (5 \times 12^1) + (2 \times 1)$$

$$\text{one hundred forty} = E8_{twelve} = (E \times 12^1) + (8 \times 1)$$

$$\text{two hundred sixty-six} = 1T2_{twelve} = (1 \times 12^2) + (T \times 12^1) + (2 \times 1)$$

Although this addition of symbols to the system for writing numerals may seem inconvenient, base twelve has commercial uses. Grouping by twelves lends itself to the business world in activities such as buying eggs by the dozen and pencils by the gross (twelve x twelve).

Base Two

Another familiar system of numeration uses two as a base. Commonly called the binary system it uses only two digits, 0 and 1.

See if you can verify the value of the following binary numerals. Remember, grouping is by two's in the binary system.

$$\text{fifteen} = 1111_{two} = (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$

$$\text{twenty-two} = 10110_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)$$

$$\text{one hundred forty} = 10001100_{two}$$

Because the binary system uses only two symbols it is particularly adapted to the "on" and "off" switch requirements of modern computers. However, because of the limited number of digits in the system, it is hard to draw a good comparison to the decimal system.

Summary

The following chart is helpful in understanding better the numeral sequence for place value numeration systems with different bases.

Twelve	Ten	Eight	Seven	Five	Four	Three	Two
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	10
3	3	3	3	3	3	10	11
4	4	4	4	4	10	11	100
5	5	5	5	10	11	12	101
6	6	6	6	11	12	20	110
7	7	7	10	12	13	21	111
8	8	10	11	13	20	22	1000
9	9	11	12	14	21	100	1001
T	10	12	13	20	22	101	1010
E	11	13	14	21	23	102	1011
10	12	14	15	22	30	110	1100
11	13	15	16	23	31	111	1101
12	14	16	20	24	32	112	1110
13	15	17	21	30	33	120	1111
14	16	20	22	31	100	121	10000
15	17	21	23	32	101	122	10001
16	18	22	24	33	102	200	10010
17	19	23	25	34	103	201	10011
18	20	24	26	40	110	202	10100

Note that the base numeral always appears as 10 when written in that particular base system. Similarly, the second power of the base (base × base) is indicated by 100 in that base.

It is important that teachers and their students gain an understanding of the structure of numeration systems with different bases, and that they make comparisons among the systems. Memory work is not necessary in this particular study.

Class Exercises

12. Complete the following table.

Base	Place Values			
Twelve				
Ten	thousands	hundreds	tens	ones
Eight				
Seven				
Five				
Four	sixty-fours			
Three				
Two				

13. Draw a diagram, using x's, to illustrate each of the numerals given. Circle groups of x's as indicated by the base of each numeral.

(a) 211_{three} (b) 1010_{two} (c) 113_{four}

14. For each part of Exercise 13 write the numeral in expanded notation, using exponents.

15. The diagram represents a set of seventeen objects. In each case complete the sentence given and then represent the number in the corresponding base.

x x x x x x
x x x x x
x x x
x x x

(a) Separate the set into groups of twelve.
There are ______ twelves and _____ ones.

(b) Separate the set into groups of ten.
There are ______ tens and _______ ones.

(c) Separate the set into groups of eight.
There are ______ eights and ______ ones.

(d) Separate the set into groups of seven.
There are ____ sevens and ____ ones.

(e) Separate the set into groups of five.
There are ____ fives and ____ ones.

(f) Separate the set into groups of four.
There are ____ four × fours, ____ fours and ____ ones.

(g) Separate the set into groups of three.
There are ____ three × threes, ____ threes and ____ ones.

(h) Separate the set into groups of two.
There are ____ two × two × two × twos, ____ two × two × twos, ____ two × twos, ____ twos, ____ ones.

2.4 Changing from One Number Base to Another

In Section 2.3 we learned that a particular set of objects may be grouped by tens, by sevens, or by other numbered groups greater than one. This means that we can represent the same number by different numerals when using different bases.

Convert $3E_{twelve}$ to base ten.

$$\begin{aligned} 3E_{twelve} &= (3 \times 12^1) + (E \times 1) \\ &= 36 + 11 \\ &= 47 \end{aligned}$$

Convert $23T_{twelve}$ to base ten.

$$\begin{aligned} 23T_{twelve} &= (2 \times 12^2) + (3 \times 12^1) + (T \times 1) \\ &= (2 \times 144) + (3 \times 12) + (T \times 1) \\ &= 288 + 36 + 10 \\ &= 334 \end{aligned}$$

Class Exercises

16. By means of expanded notation, convert each numeral to a base ten numeral:

(a) 231_{four}
(b) 35_{seven}
(c) 101011_{two}
(d) 212_{three}
(e) 66_{eight}
(f) 341_{five}
(g) $3T1_{twelve}$
(h) $T0_{twelve}$

Changing from Base Ten to Other Bases

You have learned how to change a numeral written in base seven to the corresponding base ten numerals. It is also possible to change from a base ten to a base seven numeral. Let us see how this is done.

In base seven, the values of the places are powers of seven.

$$\begin{aligned}7^0 &= & &1,\\ 7^1 &= 7 &&= 7,\\ 7^2 &= 7\times 7 &&= 49,\\ 7^3 &= 7\times 7\times 7 &&= 343,\\ 7^4 &= 7\times 7\times 7\times 7 &&= 2401,\end{aligned}$$

and so on.

Suppose that we wish to change from the base ten decimal numeral 12 to a corresponding base seven numeral. Instead of actually grouping marks, we first think of groups or powers of seven. What is the largest power of seven which is contained in 12 ? Is 7^1 (7) the largest? How about 7^2 (49) or 7^3 (343) ? Only 7^1 and 1 are small enough to be contained in 12. Hence, 7^1 is the largest power of seven included in 12. To find how many sevens are contained in 12, we divide:

$$\begin{array}{r} 1\\ 7\,\overline{)\,12}\\ \underline{7}\\ 5\end{array}$$

The quotient 1 means that there is 1 seven contained in 12; the remainder 5 means that there are 5 ones left over. Now we are able to write the base ten numeral 12 as a base seven numeral.

$$\begin{aligned}12 &= (1\times 7) + (5\times 1)\\ &= 15_{\text{seven}}.\end{aligned}$$

Consider next the decimal numeral 64. To write this in base seven, we first think of the largest power of seven contained in 64. This is 7^2 or 49. Thus, we can write:

$$64 = (\underline{?}\times 49) + (\underline{?}\times 7) + (\underline{?}\times 1).$$

The first division shown enables us to replace the first blank space with 1. However, the remainder 15 still contains the first power of seven. A second division, this time by 7, gives a remainder of 1. We may now complete the sentence as:

$$\begin{array}{r} 1\\ 49\,\overline{)\,64}\\ \underline{49}\\ 15\end{array}\qquad\begin{array}{r} 2\\ 7\,\overline{)\,15}\\ \underline{14}\\ 1\end{array}$$

$$\begin{aligned}64 &= (1\times 49) + (2\times 7) + (1\times 1)\\ &= (1\times 7^2) + (2\times 7^1) + (1\times 1)\\ &= 121_{\text{seven}}.\end{aligned}$$

Let us change the decimal numeral 524 to a base seven numeral. Recalling again the powers of seven, we select 343 as the largest power contained in 524. Dividing by this power gives a quotient of 1 and a remainder of 181. Next, we divide this remainder by the largest remaining power of seven, 49. We continue to divide each new remainder by decreasing powers of seven. Now we are able to write, in expanded form, the sentence:

$$\begin{array}{rl} & 1 \\ 343 & \overline{)524} \\ & \underline{343} \rightarrow (1 \times 343) \\ & 181 \end{array}$$

$$\begin{array}{rl} & 3 \\ 49 & \overline{)181} \\ & \underline{147} \rightarrow (3 \times 49) \\ & 34 \end{array}$$

$$\begin{array}{rl} & 4 \\ 7 & \overline{)34} \\ & \underline{28} \rightarrow (4 \times 7) \\ & 6 \rightarrow (6 \times 1) \end{array}$$

$$\begin{aligned} 524 &= (1 \times 343) + (3 \times 49) + (4 \times 7) + (6 \times 1) \\ &= (1 \times 7^3) + (3 \times 7^2) + (4 \times 7^1) + (6 \times 1) \\ &= 1346_{\text{seven}}. \end{aligned}$$

In changing base ten numerals to base seven, first select the largest place value of base seven (power of seven) contained in the number. Divide the number by this power of seven and find the quotient and remainder. The quotient is the first digit in the base seven numeral. Divide the remainder by the next smaller power of seven and this quotient is the second digit. Continue to divide each new remainder by each succeedingly smaller power of seven until all the remaining digits in the base seven numeral are found.

In converting base ten numerals to base four numerals, the same procedure is used this time with division being powers of four. As an example, change the decimal numeral 53 to a base four numeral. The powers of four are:

$$4^0 = 1, \quad 4^1 = 4, \quad 4^2 = 16, \quad 4^3 = 64, \quad \text{and so on.}$$

These, of course, become the place values of base four numerals. Successive division by decreasing powers of four give the results shown. Thus, we may write:

$$\begin{array}{rl} & 3 \\ 16 & \overline{)53} \\ & \underline{48} \rightarrow (3 \times 16) \\ & 5 \end{array}$$

$$\begin{array}{rl} & 1 \\ 4 & \overline{)5} \rightarrow (1 \times 4) \\ & \underline{4} \\ & 1 \rightarrow (1 \times 1) \end{array}$$

$$\begin{aligned} 53 &= (3 \times 16) + (1 \times 4) + (1 \times 1) \\ &= (3 \times 4^2) + (1 \times 4^1) + (1 \times 1) \\ &= 311_{\text{four}}. \end{aligned}$$

As another example,

$$\begin{aligned} 113 &= (1 \times 64) + (3 \times 16) + (0 \times 4) + (1 \times 1) \\ &= (1 \times 4^3) + (3 \times 4^2) + (0 \times 4^1) + (1 \times 1) \\ &= 1301_{\text{four}}. \end{aligned}$$

To convert base ten numerals to base five numerals, study these examples:

$$\begin{aligned} 105 &= (4 \times 25) + (1 \times 5) + (0 \times 1) \\ &= (4 \times 5^2) + (1 \times 5^1) + (0 \times 1) \\ &= 410_{\text{five}}. \end{aligned}$$

$$\begin{aligned} 780 &= (1 \times 625) + (1 \times 125) + (1 \times 25) + (1 \times 5) + (0 \times 1) \\ &= (1 \times 5^4) + (1 \times 5^3) + (1 \times 5^2) + (1 \times 5^1) + (0 \times 1) \\ &= 11110_{\text{five}}. \end{aligned}$$

By divisions such as we have performed in earlier examples, you may verify these conversions.

Thus, by the use of the expanded notation, we are able to do conversions from base ten numerals to numerals in other bases and vice versa.

Class Exercises

17. Change these base ten numerals to base seven numerals:

(a) 509
(b) 31
(c) 28
(d) 350
(e) 2500
(f) 686
(g) 6
(h) 2400

18. Change these base ten numerals to the base indicated:

(a) $86 = \underline{\ ?\ }_{\text{four}}$
(b) $21 = \underline{\ ?\ }_{\text{four}}$
(c) $250 = \underline{\ ?\ }_{\text{four}}$
(d) $250 = \underline{\ ?\ }_{\text{five}}$
(e) $124 = \underline{\ ?\ }_{\text{five}}$

2.5 Just For Fun

1. People who work with high speed computers sometimes find it easier to express numbers in the octal, or eight, system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

2. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

   What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounce to 15 ounces? From 1 ounce to 31 ounces?

   This problem is related to the weighing problem posed in the introduction of this book. It is also related to the binary numeration system. Do you see the relationship?

3. (a) Convert the base five numeral 31.42_{five} to a base ten numeral.

   (b) Convert the base ten decimal numeral 28.76 to a base five numeral.

4. Students will enjoy the following card trick which depends upon the use of base two thinking.

A		B		C		D	
1	9	2	10	4	12	8	12
3	11	3	11	5	13	9	13
5	13	6	14	6	14	10	14
7	15	7	15	7	15	11	15

Directions: Make a set of cards as shown. Tell a person to think of a number between 1 and 15 and then to tell you on which card (or cards) it appears.

You can tell him the number by getting the sum of the first number on every card named.

Example: The number 13 is shown on cards A, C, and D. Add 1, 4, and 8 to find the number.

A fuller discussion and extension of this card device may be found on page 41 of the Teacher's Commentary for Junior High School, Volume 1, Part 1.

An interesting discussion and activities on card punching appear in a free booklet, Mathematics In Action, which is obtainable from the Institute of Life Insurance, 277 Park Avenue, New York, N.Y.

5. In the marble problem posed in the introduction of this book, the number of weighings required to locate the heavy marble among a number of marbles was determined. Complete the following table.

Number of Marbles	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Number of Weighings	0	1	1	2	2			2																				

Now write the base three numerals for the numbers from 1 to 28. Do you see a pattern between the number of weighings required as listed in the table and the corresponding base three numerals representing the numbers of marbles weighed. Using this pattern, find how many weighings it would take to do a corresponding problem with 57 marbles.

Chapter Exercises

1. Write the following base ten numbers using
   (1) Egyptian numerals
   (2) Base five numerals
   (3) Base seven numerals

   (a) 19 (b) 53 (c) [illegible] (d) [illegible]

2. Write each numeral below in expanded notation, using the exponential form.

   (a) 100_{five} (e) 17_{eight}
   (b) 1110_{two} (f) T_{twelve}
   (c) $[illegible]_{three}$ (g) 100_{ten}
   (d) 110_{four} (h) 100_{seven}

3. In the base given, represent one less than each number represented in Exercise 2.

4. Suppose you are paying each amount of money listed in the left column. Rules of the game are (1) that you use only quarters, nickels, and pennies for payment, and (2) that you use the smallest number of coins. Complete the table.

Amount of Money	Number of Quarters	Number of Nickels	Number of Pennies	Base five Numeral
Example: 29 cents	1	0	4	104_{five}
a. 37 cents				
b. 24 cents				
c. 99 cents				
d. 15 cents				
e. 63 cents				
f. $1.24 (124 cents)				

5. Make up a base five system using 0, /, ∠, △, and □, for symbols representing the numbers 0, 1, 2, 3, 4. Represent the base ten numbers 1 - 25 in your system.

6. Represent each of the given numerals as a base ten decimal numeral.

(a) 2134_{five}　　(c) $T5E_{twelve}$

(b) 364_{seven}　　(d) 10031_{four}

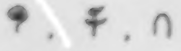
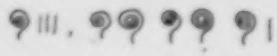
7. Represent each decimal numeral in the base indicated.

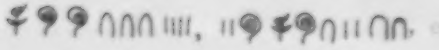
(a) $713 = \underline{\ ?\ }_{six}$　　(c) $155 = \underline{\ ?\ }_{twelve}$

(b) $44 = \underline{\ ?\ }_{two}$　　(d) $333 = \underline{\ ?\ }_{four}$

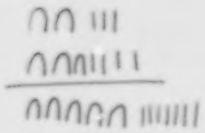
8. Represent 214_{five} as a base eight numeral.

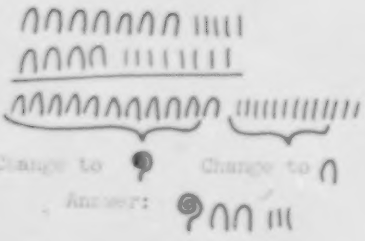
Answers to Class Exercises

1. (a) 
(b) 

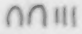
2.  etc.

3. First, arrange the symbols in the same order:

(a) 

(b) 

4. It becomes a series of additions.

Add  5 times.

5. (a) $(4 \times 10^2) + (3 \times 10^1) + (6 \times 1)$

(b) $(5 \times 10^3) + (4 \times 10^2) + (0 \times 10^1) + (3 \times 1)$

(c) $(3 \times 10^4) + (3 \times 10^3) + (9 \times 10^2) + (6 \times 10^1) + (7 \times 1)$

(d) $(5 \times 10^6)+(2 \times 10^5)+(5 \times 10^4)+(6 \times 10^3)+(8 \times 10^2)+(9 \times 10^1)+(8 \times 1)$

6.

A Number Decimal Numeral	B Product Expression with Repeated Factors	C Exponential Form	D Powers of Ten
(a) 100	10 × 10	10^2	Second
(b) 10,000	10×10×10×10	10^4	Fourth
(c) 100,000	10×10×10×10×10	10^5	Fifth
(d) 1,000,000	10×10×10×10×10×10	10^6	Sixth
(e) 100,000,000	10×10×10×10×10×10×10×10	10^8	Eighth

7. (a) 400,000,000
(c) 500,005
(b) 3,020,500
(d) 4,006

8. $13 \times 10^{[illegible]}$, [illegible] $\times 13 \times 10^{[illegible]}$

9. 10^{100}

10. (a) [illegible]
(c) 32_{five}
(b) [illegible]
(d) four, two, base five

11. $\underline{4}$ $\underline{4}$ $\underline{44}_{five}$,

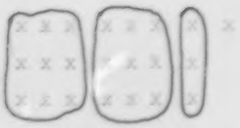
$\underline{3}$ $\underline{2}$ $\underline{32}_{five}$,

$\underline{3}$ $\underline{0}$ $\underline{30}_{five}$.

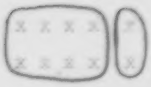
12.

Base	Place Value			
Twelve	One thousand seven hundred twenty-eight	One hundred forty fours	Twelves	Ones
Ten	Thousands	Hundreds	Tens	Ones
Eight	Five hundred twelves	Sixty fours	Eights	Ones
Seven	Three hundred forty - threes	Forty nines	Sevens	Ones
Five	One hundred twenty fives	Twenty fives	Fives	Ones
Four	Sixty-fours	Sixteens	Fours	Ones
Three	Twenty sevens	Nines	Threes	Ones
Two	Eights	Fours	Twos	Ones

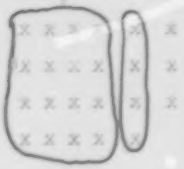
13. (a)



(b)



(c)



14. (a) $211_{three} = (2 \times 3^2) + (1 \times 3^1) + (1 \times 3^0)$

(b) $1010_{two} = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)$

(c) $113_{four} = (1 \times 4^2) + (1 \times 4^1) + (3 \times 4^0)$

15. (a) 15_{twelve}

(b) 17_{ten}

(c) 21_{eight}

(d) 23_{seven}

(e) 32_{five}

(f) 101_{four}

(g) 122_{three}

(h) 10001_{two}

16. (a) $231_{four} = (2 \times 4^2) + (3 \times 4) + (1 \times 1)$
$= 32 + 12 + 1$
$= 45_{ten}$

(b) $35_{seven} = (3 \times 7) + (5 \times 1)$
$= 21 + 5$
$= 26_{ten}$

(c) $101011_{two} = (1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2) + (1 \times 1)$
$= 32 + 0 + 8 + 0 + 2 + 1$
$= 43_{ten}$

(d) $212_{three} = (2 \times 3^2) + (1 \times 3) + (2 \times 1)$
$= 18 + 3 + 2$
$= 23_{ten}$

(e) $66_{eight} = (6 \times 8) + (6 \times 1)$
$= 48 + 6$
$= 54_{ten}$

(f) $341_{five} = (3 \times 5^2) + (4 \times 5) + (1 \times 1)$
$= 75 + 20 + 1$
$= 96_{ten}$

(g) $3T1_{twelve} = (3 \times 12^2) + (T \times 12) + (1 \times 1)$
$= 432 + 120 + 1$
$= 553_{ten}$

(h) $T0_{twelve} = (T \times 12) + (0 \times 1)$
$= 120 + 0$
$= 120_{seven}$

17. (a) $509_{ten} = 1325_{seven}$

$$\begin{array}{r} 1 \\ 343\overline{)509} \\ \underline{343} \\ 166 \end{array} \qquad \begin{array}{r} 3 \\ 49\overline{)166} \\ \underline{147} \\ 19 \end{array} \qquad \begin{array}{r} 2 \\ 7\overline{)19} \\ \underline{14} \\ 5 \end{array}$$

(b) $31_{ten} = 43_{seven}$

$$\begin{array}{r} 4 \\ 7\overline{)31} \\ \underline{28} \\ 3 \end{array}$$

(c) $28_{ten} = 40_{seven}$

(d) $350_{ten} = 1010_{seven}$

$$\begin{array}{r} 1 \\ 343\overline{)350} \\ \underline{343} \\ 7 \end{array} \qquad \begin{array}{r} 0 \\ 49\overline{)7} \\ \underline{0} \\ 7 \end{array} \qquad \begin{array}{r} 1 \\ 7\overline{)7} \\ \underline{7} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)0} \\ \underline{0} \\ 0 \end{array}$$

(e) $2500_{ten} = 10201_{seven}$

$$\begin{array}{r} 1 \\ 2401\overline{)2500} \\ \underline{2401} \\ 99 \end{array} \qquad \begin{array}{r} 0 \\ 343\overline{)99} \\ \underline{0} \\ 99 \end{array} \qquad \begin{array}{r} 2 \\ 49\overline{)99} \\ \underline{98} \\ 1 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)1} \\ \underline{0} \\ 1 \end{array} \qquad \begin{array}{r} 1 \\ 1\overline{)1} \\ \underline{1} \\ 0 \end{array}$$

(f) $686_{ten} = 2000_{seven}$

$$\begin{array}{r} 2 \\ 343\overline{)686} \\ \underline{686} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 49\overline{)0} \\ \underline{0} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)0} \\ \underline{0} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 1\overline{)0} \\ \underline{0} \\ 0 \end{array}$$

(g) $6_{ten} = 6_{seven}$

(h) $2400_{ten} = 6666_{seven}$

$$\begin{array}{r} 6 \\ 343\overline{)2400} \\ \underline{2058} \\ 342 \end{array} \qquad \begin{array}{r} 6 \\ 49\overline{)342} \\ \underline{294} \\ 48 \end{array} \qquad \begin{array}{r} 6 \\ 7\overline{)48} \\ \underline{42} \\ 6 \end{array}$$

18. (a) $85_{ten} = 1111_{four}$

$$\begin{array}{r} 1 \\ 64\overline{)85} \\ \underline{64} \\ 21 \end{array} \qquad \begin{array}{r} 1 \\ 16\overline{)21} \\ \underline{16} \\ 5 \end{array} \qquad \begin{array}{r} 1 \\ 4\overline{)5} \\ \underline{4} \\ 1 \end{array}$$

(b) $21_{ten} = 111_{four}$

(c) $250_{ten} = 3322_{four}$

$$\begin{array}{r} 3 \\ 64\overline{)250} \\ \underline{192} \\ 58 \end{array} \qquad \begin{array}{r} 3 \\ 16\overline{)58} \\ \underline{48} \\ 10 \end{array} \qquad \begin{array}{r} 2 \\ 4\overline{)10} \\ \underline{8} \\ 2 \end{array}$$

(d) $250_{ten} = 2000_{five}$

$$\begin{array}{r} 2 \\ 125\overline{)250} \\ \underline{250} \\ 0 \end{array}$$

(e) $124_{ten} = 444_{five}$

$$\begin{array}{r} 4 \\ 25\overline{)124} \\ \underline{100} \\ 24 \end{array} \qquad \begin{array}{r} 4 \\ 5\overline{)24} \\ \underline{20} \\ 4 \end{array}$$

Chapter 3

COMPUTATION IN BASES OTHER THAN TEN

Introduction

One purpose of Chapter 2 was to reveal the structure of the decimal system by examining similar structures in other systems; for instance, base five. In this chapter, intended as a sequel to Chapter 2, operations are performed in some of the bases. Such computation has two values for seventh grade students: (1) It provides practice in the fundamental operations free from repetition of the base ten work in elementary school mathematics; and, (2) it strengthens both their skills in and understandings of operations with decimal numerals rather than just supplying more computation.

Emphasis here is placed on base five computation, with suggestions and problems for other bases. The teacher may wish to refer to the base seven study presented in Chapter 2 of the Student's Text, Mathematics for Junior High School, Volume 1, Part 1; and in Studies in Mathematics, Volume VI. Addition and multiplication facts should not be memorized but the process and understanding should be stressed. To this end, once the addition and multiplication tables have been developed, they should be made available to the students for reference when doing computation. Teachers should again be cautioned against spending too much time on work with other number bases. We would hope that poorer students may gain some understanding of the algorithms and good students become proficient in computations. We do not want teachers to drill all students to the point of boredom for better students, and frustration for poorer students.

3.1 Addition

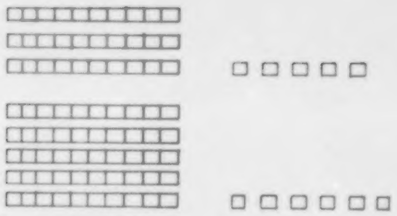
We are going to explore addition in several bases other than ten, with emphasis on base five. Suppose for the moment that we are "moon people" and that we count by fives because there are five fingers on one hand. Students may begin addition in base five by finding sums of simple problems such as:

$3_{\text{five}} + 1_{\text{five}}$ (4_{five})
$2_{\text{five}} + 0_{\text{five}}$ (2_{five})
$3_{\text{five}} + 2_{\text{five}}$ (10_{five})
$2_{\text{five}} + 4_{\text{five}}$ (11_{five})

Before we can go farther into addition in base five, we shall consider what the addition algorithm really means in base ten. An algorithm is a way of recording the thought processes. In base ten addition let us talk about:

$$\begin{array}{rl} 35 = & 3 \text{ tens} + 5 \text{ ones} \\ \underline{+\ 56} = & \underline{5 \text{ tens} + 6 \text{ ones}} \\ & 8 \text{ tens} + 11 \text{ ones} = 8 \text{ tens} + 1 \text{ ten} + 1 \text{ one} = \\ & \qquad 9 \text{ tens} + 1 \text{ one} = 91. \end{array}$$

To add 35 and 56 it is impractical to draw 35 x's and 56 x's, group them in tens and ones, and then count the number of tens and the number of ones, even though this is what we really mean by addition of whole numbers. To avoid this cumbersome method, we break the problem down into several small problems as indicated in this figure.



Let us combine the small boxes representing ones: 5 + 6 = 11. Combining the larger boxes representing tens, we have: 3 + 5 = 8. Now 11 small boxes is the same as 1 large box and 1 small box. The total, then, is 9 large boxes + 1 small box. This sum is recorded as a written numeral in the addition problem given above.

We may think of any addition problem in this way. In base ten it involves small boxes (1's), large boxes (10's), giant size boxes (100's), economy size boxes (1,000's), family size boxes (10,000's), and so on. Thinking this way in the last problem, we did not need to know the combination 35 + 56; we only needed to know 5 + 6 and 3 + 5. If this line of thought is pursued, one is soon convinced that any addition problem, base ten, may be done if one knows the entries in the table of addition combinations. Likewise, addition in any base can be performed given the table of addition combinations in that base.

Our algorithms exist to eliminate this physical approach to problems. When we perform, mechanically, the addition 35 + 56 = 91, we are indicating this procedure without thinking every step through each time as we did earlier.

Let us now make a base five table of addition. The study of subsequent chapters will reveal several properties which allow us to extend the basic combinations to any number in the system. Making an addition table for base five identifies the twenty-five basic addition combinations to be used in computation.

In teaching a seventh grade class, the addition table for base five can be developed in class. This could be accomplished by preparing the array and inserting only a portion of the entries; students can assist in determining the appropriate entries for the remaining spaces.

Addition Table, Base Five

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Now let us return to addition in base five. A technique teachers may use with their classes is to demonstrate, with objects such as stars, the following additions in base five.

★★★★ + ★★★ = ★★★★★ + ★★

$$4_{\text{five}} + 3_{\text{five}} = 10_{\text{five}} + 2_{\text{five}} = 12_{\text{five}}$$

The concept of grouping by fives is to be emphasized.

If the expanded notation developed earlier is followed in thinking through base five addition, then we have: (using the addition table)

$$\begin{array}{rl}
22_{\text{five}} & = 2 \text{ fives} + 2 \text{ ones} \\
\underline{+\ 14_{\text{five}}} & = \underline{1 \text{ five} + 4 \text{ ones}} \\
 & 3 \text{ fives} + 11 \text{ ones} \\
 & = 3 \text{ fives} + (1 \text{ five} + 1 \text{ one}) \\
 & = (3 \text{ fives} + 1 \text{ five}) + 1 \text{ one} \\
 & = 4 \text{ fives} + 1 \text{ one} \\
 & = 41_{\text{five}}.
\end{array}$$

Observe that the notation "11 ones" is base five notation being used in a base five problem. Here are several more examples expressed in somewhat simplified form.

$$\begin{array}{rl} 21_{\text{five}} & = 2 \text{ fives} + 1 \text{ one} \\ \underline{+\ 24_{\text{five}}} & = \underline{2 \text{ fives} + 4 \text{ ones}} \\ & \quad 4 \text{ fives} + 10 \text{ ones} \\ & = (4 \text{ fives} + 1 \text{ five}) + 0 \text{ ones} \\ & = 10 \text{ fives} + 0 \text{ ones} \\ & = 1 \text{ five} \times \text{five} + 0 \text{ fives} + 0 \text{ ones} = 100_{\text{five}}. \end{array}$$

$$\begin{array}{rl} 223_{\text{five}} & = 2 \text{ five} \times \text{fives} + 2 \text{ fives} + 3 \text{ ones} \\ \underline{+\ 243_{\text{five}}} & = \underline{2 \text{ five} \times \text{fives} + 4 \text{ fives} + 3 \text{ ones}} \\ & \quad 4 \text{ five} \times \text{fives} + 11 \text{ fives} + 11 \text{ ones} \\ & = 1 \text{ five} \times \text{five} \times \text{five} + 0 \text{ five} \times \text{fives} + 2 \text{ fives} + 1 \text{ one} \\ & = 1021_{\text{five}}. \end{array}$$

In each of these cases it has been necessary to "regroup". Regrouping in base five means renaming

10_{five} ones as 1 five
10_{five} fives as 1 five × five
10_{five} five × fives as 1 five × five × five.

This corresponds to the base ten regrouping which means renaming

10 ones as 1 ten
10 tens as 1 ten × ten
10 ten × tens as 1 ten × ten × ten.

In any base regrouping involves an exchange between place value positions as shown; groups correspond to the powers of the base being used. Thus,

10_{five} means 1 group of fives, while
10_{seven} means 1 group of sevens, and
10_{three} means 1 group of threes.

The term regrouping used here is the same process often referred to in base ten as "carrying." When teaching addition in these and other bases it is helpful to have students construct addition tables for easy reference.

Addition Table, Base Seven

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

Addition Table, Base Three

+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

A special addition with regrouping occurs with denominate numbers. Tables of measure determine the grouping. Several examples show that regrouping is used with addition of denominate numbers.

$$\begin{array}{rl} & 3421 \text{ feet} \\ + & 4890 \text{ feet} \\ \hline & 8311 \text{ feet} = 1 \text{ mile} + 3031 \text{ feet} \end{array}$$

$$\begin{array}{rllll} & 1 \text{ m.} & 8 \text{ dm.} & 3 \text{ cm.} & 5 \text{ mm.} \\ + & 5 \text{ m.} & 5 \text{ dm.} & 8 \text{ cm.} & 3 \text{ mm.} \\ \hline & 6 \text{ m.} & 13 \text{ dm.} & 11 \text{ cm.} & 8 \text{ mm.} \\ = & 7 \text{ m.} & 4 \text{ dm.} & 1 \text{ cm.} & 8 \text{ mm.} \end{array}$$

$$\begin{array}{rlll} & 4 \text{ grams} & 2 \text{ decigrams} & 8 \text{ centigrams} \\ + & 2 \text{ grams} & 7 \text{ decigrams} & 7 \text{ centigrams} \\ \hline & 6 \text{ grams} & 9 \text{ decigrams} & 15 \text{ centigrams} \\ = & 7 \text{ grams} & 0 \text{ decigrams} & 5 \text{ centigrams} \end{array}$$

$$\begin{array}{rlll} & 3 \text{ weeks} & 4 \text{ days} & 18 \text{ hours} \\ + & 2 \text{ weeks} & 6 \text{ days} & 6 \text{ hours} \\ \hline & 5 \text{ weeks} & 10 \text{ days} & 24 \text{ hours} \\ = & 6 \text{ weeks} & 4 \text{ days} & 0 \text{ hours} \end{array}$$

Addition in base five or in any other base may be checked by renaming the numerals in decimal notation and adding. For example:

$$\begin{array}{rcr} \underline{\text{Base Five}} & & \underline{\text{Base Ten}} \\ 21_{\text{five}} & \longleftrightarrow & 11 \\ + \; 24_{\text{five}} & \longleftrightarrow & + \; 14 \\ \hline 100_{\text{five}} & \longleftrightarrow & 25_{\text{ten}} \end{array}$$

The addition of two numbers is represented below in four different bases. Verify that each is the same problem simply expressed in a different base from the others.

Base Ten		Base Twelve		Base Eight		Base Three
$\begin{array}{r} 299 \\ +\ 27 \\ \hline 326_{ten} \end{array}$	$\longleftrightarrow$	$\begin{array}{r} 20E_{twelve} \\ +\ 23_{twelve} \\ \hline 232_{twelve} \end{array}$	$\longleftrightarrow$	$\begin{array}{r} 453_{eight} \\ +\ 33_{eight} \\ \hline 506_{eight} \end{array}$	$\longleftrightarrow$	$\begin{array}{r} 32002_{three} \\ +\ 1000_{three} \\ \hline 33002_{three} \end{array}$

Addition in bases other than ten is included in a seventh grade mathematics program because it helps to clarify addition in decimal notation while at the same time illustrating certain number properties. The words "regroup" or "rename" are found in many commercial textbooks; they are preferred by most elementary school teachers over the term "carry," which they replace. An application of regrouping occurs in addition with denominate numbers, as seen in the examples given in this section.

Class Exercises

1. Complete the following table showing the basic addition combinations for base eight.

Addition Table, Base Eight

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

2. Add the following, noting the base in which each is written.

(a) $\begin{array}{r} 22_{five} \\ +\ 13_{five} \\ \hline \end{array}$

(c) $\begin{array}{r} 177_{eight} \\ +\ 101_{eight} \\ \hline \end{array}$

(b) $\begin{array}{r} 43_{five} \\ +\ 14_{five} \\ \hline \end{array}$

(d) $\begin{array}{r} 321_{eight} \\ +\ 275_{eight} \\ \hline \end{array}$

3. Check each addition in Exercise 2 by first changing the numerals to base ten.

4. Add as indicated and check using base five numerals.

(a) $\begin{array}{r} 75_{\text{ten}} \\ +\ 313_{\text{ten}} \\ \hline \end{array}$ (b) $\begin{array}{r} 39_{\text{ten}} \\ +\ 104_{\text{ten}} \\ \hline \end{array}$

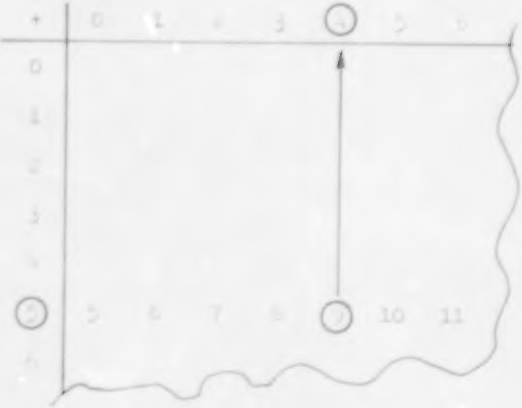
5. Add:

(a) $\begin{array}{r} 11_{\text{two}} \\ +\ 11_{\text{two}} \\ \hline \end{array}$ (b) $\begin{array}{r} 32_{\text{five}} \\ 32_{\text{five}} \\ 32_{\text{five}} \\ 32_{\text{five}} \\ +\ 32_{\text{five}} \\ \hline \end{array}$ (c) $\begin{array}{r} 43_{\text{five}} \\ 43_{\text{five}} \\ 43_{\text{five}} \\ 43_{\text{five}} \\ +\ 43_{\text{five}} \\ \hline \end{array}$ (d) $\begin{array}{r} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ +\ 24 \\ \hline \end{array}$ (e) $\begin{array}{r} 26_{\text{seven}} \\ 26_{\text{seven}} \\ 26_{\text{seven}} \\ 26_{\text{seven}} \\ 26_{\text{seven}} \\ 26_{\text{seven}} \\ +\ 26_{\text{seven}} \\ \hline \end{array}$

Explain the connection between the addends and the sum for each part.

3.2 Subtraction

Most people learn to subtract in base ten by practicing certain subtraction combinations long enough to become thoroughly familiar with them. Suppose we pretend for the moment that you do not know the answer to the subtraction 9 - 5. The answer can be found in the base ten addition table. The question you really need to answer is "What number, when added to 5, yields 9?"



The table shows that addition may be used to solve subtraction problems. Since $4 + 5 = 9$, it is implied that $9 - 5 = 4$. Subtraction is the inverse operation of addition. In a later chapter the concept of inverse operations will be discussed in more detail.

Using only the base five addition table verify that each of these subtraction problems is correct:

$$10_{five} - 4_{five} = 1_{five}$$

$$11_{five} - 3_{five} = 3_{five}$$

$$13_{five} - 4_{five} = 4_{five}$$

A simple subtraction problem in base five, where no regrouping is necessary, is shown below.

$$\begin{array}{rl} 34_{five} & = 3 \text{ fives} + 4 \text{ ones} \\ \underline{-\ 21_{five}} & = \underline{2 \text{ fives} + 1 \text{ one}} \\ & \ 1 \text{ five} + 3 \text{ ones} = 13_{five} \end{array}$$

More difficult subtraction problems involve regrouping. Just as we reviewed regrouping in addition, let us look at an example of regrouping in base ten subtraction.

$$\begin{array}{rlll} 55 & = 5 \text{ tens} + 5 \text{ ones} & = 4 \text{ tens} + 15 \text{ ones} & \\ \underline{-\ 27} & = \underline{2 \text{ tens} + 7 \text{ ones}} & = \underline{2 \text{ tens} + \ 7 \text{ ones}} & \\ & & \ 2 \text{ tens} + \ 8 \text{ ones} & = 28 \end{array}$$

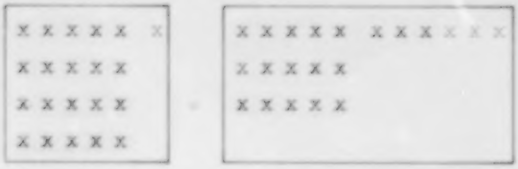
Let us consider a similar subtraction problem using base five numerals. Notice how the logic of the base ten subtraction process using regrouping has been followed.

$$\begin{array}{rlll} 41_{five} & = 4 \text{ fives} + 1 \text{ one} & = 3 \text{ fives} + 11 \text{ ones} & \\ \underline{-\ 23_{five}} & = \underline{2 \text{ fives} + 3 \text{ ones}} & = \underline{2 \text{ fives} + \ 3 \text{ ones}} & \\ & & \ 1 \text{ five} + \ 3 \text{ ones} & = 13_{five} \end{array}$$

The diagram below illustrates how the same problem can be represented by showing regrouping as the partitioning of a set. This method is frequently helpful in explaining the operation of subtraction to students.

Base Five

$$\begin{array}{r} 41_{five} \\ -\ 23_{five} \\ \hline 13_{five} \end{array}$$



Another example of subtraction using base five numerals is shown here. Again note the use of regrouping.

$$\begin{array}{rl} 100_{five} & = 1 \text{ five} \times \text{five} + 0 \text{ fives} + 0 \text{ ones} = 4 \text{ fives} + 10 \text{ ones} \\ -\ 23_{five} & \qquad\qquad\qquad 2 \text{ fives} + 3 \text{ ones} = 2 \text{ fives} + \ 3 \text{ ones} \\ \hline & \qquad\qquad\qquad\qquad\qquad\qquad 2 \text{ fives} + \ 2 \text{ ones} = 22_{five} \end{array}$$

Following a similar procedure, teachers may guide students through subtraction problems in other bases. For example:

$$\begin{array}{rl} 21_{three} & = 2 \text{ threes} + 1 \text{ one} \ = 1 \text{ three} + 11 \text{ ones} \\ -\ 12_{three} & = 1 \text{ three} \ + 2 \text{ ones} = 1 \text{ three} + \ 2 \text{ ones} \\ \hline & \qquad\qquad\qquad\qquad\qquad\qquad 2 \text{ ones} = 2_{three} . \end{array}$$

Note here that the 2 threes and 1 one were regrouped as 1 three and 11 ones in base three notation.

Subtraction in any base may be checked by changing the numerals to decimal notation and then performing the corresponding subtraction with base ten numerals. As an alternative procedure, teachers may prefer the addition method of checking subtraction problems. This procedure gives more practice in using notation in bases other than ten. Each of the subtraction problems of this section is checked here using the corresponding addition.

Problem:

$$\begin{array}{r} 55 \\ -\ 27 \\ \hline 28 \end{array} \quad \begin{array}{r} 41_{five} \\ -\ 23_{five} \\ \hline 13_{five} \end{array} \quad \begin{array}{r} 100_{five} \\ -\ 23_{five} \\ \hline 22_{five} \end{array} \quad \begin{array}{r} 21_{three} \\ -\ 12_{three} \\ \hline 2_{three} \end{array}$$

Check:

$$\begin{array}{r} 28 \\ +\ 27 \\ \hline 55 \end{array} \quad \begin{array}{r} 13_{five} \\ +\ 23_{five} \\ \hline 41_{five} \end{array} \quad \begin{array}{r} 22_{five} \\ +\ 23_{five} \\ \hline 100_{five} \end{array} \quad \begin{array}{r} 2_{three} \\ +12_{three} \\ \hline 21_{three} \end{array}$$

Just as with addition, the subtraction of denominate numbers provides opportunities for students to develop skill in regrouping numbers used in expressing measurements. This functional aspect of regrouping is related to the study of measurement in Chapter 1[illegible]. Here is an example of subtraction with denominate numbers:

$$\begin{array}{rcrcr} 6\text{ m. } 3\text{ cm.} & = & 5\text{ m. } 103\text{ cm.} & = & 5\text{ m. } 102\text{ cm. } 10\text{ mm.} \\ \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} & = & \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} & = & \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} \\ & & & & 3\text{ m. } 87\text{ cm. } 7\text{ mm.} \end{array}$$

Class Exercises

6. Perform the indicated subtractions, noting the base in which each is written.

(a) $\begin{array}{r} 37_{twelve} \\ \underline{-\ 11_{twelve}} \end{array}$ (c) $\begin{array}{r} [illegible]_{seven} \\ \underline{-\ 1[illegible]_{seven}} \end{array}$ (e) $\begin{array}{r} [illegible]_{eight} \\ \underline{-\ 35_{eight}} \end{array}$

(b) $\begin{array}{r} 1[illegible]_{three} \\ \underline{-\ 21_{three}} \end{array}$ (d) $\begin{array}{r} [illegible]_{five} \\ \underline{-\ [illegible]3_{five}} \end{array}$ (f) $\begin{array}{r} 101_{two} \\ \underline{-\ 10_{two}} \end{array}$

7. Check the subtraction in Exercise 6([illegible]), both

(a) by verifying the subtraction in base ten, and
(b) by using addition in the original base.

8. For each pair of numerals, use =, >, or < to make a true statement.

(a) 30_{five} ______ $[illegible]_{seven}$ (c) 61_{eight} ______ 50_{ten}

(b) 11_{seven} ______ 20_{four} (d) 1111_{two} ______ $11[illegible]_{three}$

3.3 Multiplication

Multiplication is included in a study of bases other than ten because it reinforces multiplication concepts in base ten, it serves to illustrate certain number properties, and it is a vehicle for multiplication of denominate numbers.

The number of basic multiplication combinations is determined by the base. While in base five there are only 25 basic combinations, base twelve has 144, and base two has 4. Developing the multiplication table for base five identifies the basic multiplication combinations to be used in computation. Teachers may wish to develop the table as a class activity. This may be accomplished by preparing the array and inserting only part of the entries. Students can then obtain appropriate entries for blank spaces. The teacher can offer guidance as indicated in the following discussion. Once completed, the table should be left on the board for easy reference by the students when doing multiplication and division.

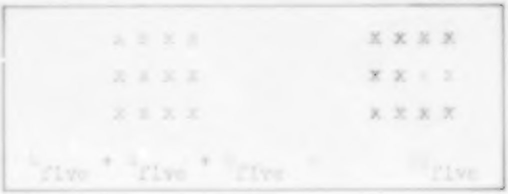
Multiplication Table, Base Five

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

An instructional technique that teachers may use to help students develop these basic multiplication facts for the table relates multiplication to repeated addition. Multiplication of counting numbers is a shortened form of addition in the special case where the addends are equal. For example,

$$3_{five} \times 4_{five}$$

may be written as $4_{five} + 4_{five} + 4_{five}$. The multiplication can thus be illustrated by objects in an array composed of three rows of 4 objects each.



The objects are grouped by fives as shown at the right. Both the sum of the addends and the product of the factors is 22_{five}. Notice that the product 22_{five} occurs two places in the multiplication table, for $4_{five} \times 3_{five}$ and for $3_{five} \times 4_{five}$. This illustrates the commutative property for the multiplication of whole numbers.

Before we consider the use of the table in multiplying with base five numerals, we should look again at multiplication in base ten. In multiplication we not only use the basic multiplication facts, but also a knowledge of the powers of ten. Although you are familiar with the multiplication algorithm, we review it here.

$$\begin{array}{rl} 732_{\text{ten}} & \\ \underline{\times\ 7_{\text{ten}}} & \\ 14 & \text{---}\ (7 \times 2) \\ 210 & \text{---}\ (7 \times 30) \\ \underline{4900} & \text{---}\ (7 \times 700) \\ 5124_{\text{ten}} & \end{array}$$

$$\begin{aligned} 7 \times 732 &= 7 \times (700 + 30 + 2) \\ &= (7 \times 700) + (7 \times 30) + (7 \times 2) \\ &= (4900) + (210) + (14) \\ &= 5124 \end{aligned}$$

In vertical form the multiplication shows partial products and how they are obtained. The multiplication shown in horizontal form uses the distributive property. Because 732_{ten} is a three-place numeral, we recognize that we have three partial products (7 × 700), (7 × 30), and (7 × 2). Taking the sum of the partial products in each case requires renaming.

Now let us examine a similar multiplication using base five numerals. Consider the product:

$$2_{\text{five}} \times 242_{\text{five}}.$$

The vertical form of multiplication again shows the partial products and how they are obtained.

$$\begin{array}{rl} 242_{\text{five}} & \\ \underline{\times\ 2_{\text{five}}} & \\ 4 & \text{---}\ (2_{\text{five}} \times 2_{\text{five}}) \\ 130 & \text{---}\ (2_{\text{five}} \times 40_{\text{five}}) \\ \underline{400} & \text{---}\ (2_{\text{five}} \times 200_{\text{five}}) \\ 1034_{\text{five}} & \end{array}$$

Sometimes it is helpful to the student to express such a problem using expanded notation so that the partial products are listed in horizontal form. This clearly shows how the basic multiplication facts from the table are used.

$$\begin{array}{rl} 242_{\text{five}} & = (2 \times \text{five}^2) + (4 \times \text{five}) + (2 \times \text{one}) \\ \underline{\times\ 2_{\text{five}}} & = \underline{\phantom{(2 \times \text{five}^2) + (4 \times \text{five}) +\ } (2 \times \text{one})} \\ & = (4 \times \text{five}^2) + (13 \times \text{five}) + (4 \times \text{one}) \\ & = (1 \times \text{five}^3) + (0 \times \text{five}^2) + (3 \times \text{five}) + (4 \times \text{one}) \\ & = 1034_{\text{five}} \end{array}$$

Multiplication in base five may be checked by changing the numerals to base ten numerals, performing the multiplication, and comparing the two products, as in the following:

$$\begin{array}{rcr} [illegible]_{five} & \longleftrightarrow & [illegible] \\ \times\ [illegible]_{five} & \longleftrightarrow & \times\ [illegible] \\ \hline 103[illegible]_{five} & & 1[illegible]_{ten}. \end{array}$$

A base system of numeration makes the computation of certain products routine. In any base, the numeral 10 names the base. For example, 10_{ten} names ten, 10_{five} names five, and 10_{twelve} names twelve. Thus, using base ten numerals, 10 × 10 is the square of the base and is written 100. Using base five numerals, 10 × 10 is also the square of the base and is written 100. Notice that in any base 100 denotes the square of the base. Similarly, 10 × 100 in any base denotes the base times the square of the base and is written 1000.

We show a use of this property in the following problem:

$$\begin{array}{rr} 324_{five} & = (3 \times five^2) + (2 \times five) + (4 \times one) \\ \times\ 10_{five} & (1 \times five) + (0 \times one) \\ \hline & \end{array}$$

Using base five notation throughout this problem can be rewritten as:

$$\begin{array}{rl} 324_{five} & = (3 \times 100_{five}) + (2 \times 10_{five}) + (4 \times 1_{five}) \\ \times\ 10_{five} & \qquad\qquad\qquad\qquad 10_{five} \\ \hline & [3 \times (100_{five} \times 10_{five})] + [2 \times (10_{five} \times 10_{five})] + [4 \times (1_{five} \times 10_{five})] \\ & = [3 \times 1000_{five}] + [2 \times 100_{five}] + [4 \times 10_{five}] \\ & = 3000_{five} + 200_{five} + 40_{five} \\ & = 3240_{five}. \end{array}$$

The same problem written in shortened form appears simply as:

$$\begin{array}{rl} 324_{five} & \\ \times\ 10_{five} & \\ \hline 40 & — (10_{five} \times 4_{five}) \\ 200 & — (10_{five} \times 20_{five}) \\ 3000 & — (10_{five} \times 300_{five}) \\ \hline 3240_{five}. & \end{array}$$

For multiplication in other bases it is desirable first to develop the tables showing the basic multiplication combinations. Base seven and base three tables are given here.

Multiplication Table, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Multiplication Table, Base Three

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

Some examples of multiplication in these bases are shown here.

$$\begin{array}{rl} 563_{\text{seven}} & \\ \times\ 5_{\text{seven}} & \\ \hline 21 & \text{---}\ (5_{\text{seven}} \times 3_{\text{seven}}) \\ 420 & \text{---}\ (5_{\text{seven}} \times 60_{\text{seven}}) \\ \underline{3400} & \text{---}\ (5_{\text{seven}} \times 500_{\text{seven}}) \\ 4141_{\text{seven}} & \end{array}$$

$$\begin{array}{rl} 342_{\text{seven}} & \\ \times\ 63_{\text{seven}} & \\ \hline 1356 & \text{---}\ (3_{\text{seven}} \times 342_{\text{seven}}) \\ \underline{30450} & \text{---}\ (60_{\text{seven}} \times 342_{\text{seven}}) \\ 32136_{\text{seven}} & \end{array}$$

$$\begin{array}{rl} 212_{\text{three}} & \\ \times\ 2_{\text{three}} & \\ \hline 11 & \text{---}\ (2_{\text{three}} \times 2_{\text{three}}) \\ 20 & \text{---}\ (2_{\text{three}} \times 10_{\text{three}}) \\ \underline{1100} & \text{---}\ (2_{\text{three}} \times 200_{\text{three}}) \\ 1131_{\text{three}} & \end{array}$$

$$\begin{array}{rl} 121_{\text{three}} & \\ \times\ 12_{\text{three}} & \\ \hline 1012 & \text{---}\ (2_{\text{three}} \times 121_{\text{three}}) \\ \underline{1210} & \text{---}\ (10_{\text{three}} \times 121_{\text{three}}) \\ 2222_{\text{three}} & \end{array}$$

$$\begin{array}{r} 201_{\text{three}} \\ \times\ 20_{\text{three}} \\ \hline 11020_{\text{three}} \end{array}$$

The multiplication at the right may also be expressed using the following development:

$$\begin{aligned} 201_{\text{three}} \times 20_{\text{three}} &= 201_{\text{three}} \times (2_{\text{three}} \times 10_{\text{three}}) \\ &= (201_{\text{three}} \times 2_{\text{three}}) \times 10_{\text{three}} \\ &= 1102_{\text{three}} \times 10_{\text{three}} \\ &= 11020_{\text{three}} . \end{aligned}$$

A special aspect of grouping in multiplication is found in obtaining products where denominate numbers are concerned. For example, "How many weeks and days are in three groups of 3 weeks and 4 days?"

$$\begin{array}{r} 3 \text{ weeks} + 4 \text{ days} \\ \times\ 3 \\ \hline 9 \text{ weeks} + 12 \text{ days} \\ = 10 \text{ weeks} + 5 \text{ days} \end{array}$$

In this example days are grouped by sevens to make weeks.

Many items are weighed by pounds and ounces. For example, John is mailing 4 packages, each of which weighs 2 pounds and 5 ounces. He needs to know the total weight of the packages.

$$\begin{array}{r} 2 \text{ pounds} + 5 \text{ ounces} \\ \times\ 4 \\ \hline 8 \text{ pounds} + 20 \text{ ounces} \\ = 9 \text{ pounds} + 4 \text{ ounces} \end{array}$$

From linear measurements we find problems such as these:

$$\begin{array}{r} 4 \text{ yds. } 2 \text{ ft. } 7 \text{ in.} \\ \times\ 4 \\ \hline 16 \text{ yds. } 8 \text{ ft. } 28 \text{ in.} \\ = 19 \text{ yds. } 1 \text{ ft. } 4 \text{ in.} \end{array} \qquad \begin{array}{r} 2 \text{ m. } 35 \text{ cm.} \\ \times\ 5 \\ \hline 10 \text{ m. } 175 \text{ cm.} \\ = 11 \text{ m. } 75 \text{ cm.} \end{array}$$

Class Exercises

9. Multiply as indicated:

(a) $\begin{array}{r} 31_{\text{five}} \\ \times\ 2_{\text{five}} \\ \hline \end{array}$ (b) $\begin{array}{r} 42_{\text{five}} \\ \times\ 4_{\text{five}} \\ \hline \end{array}$ (c) $\begin{array}{r} 234_{\text{five}} \\ \times\ 31_{\text{five}} \\ \hline \end{array}$

10. Multiply as indicated:

(a) $\begin{array}{r} 300_{\text{seven}} \\ \times\ 5_{\text{seven}} \\ \hline \end{array}$ (b) $\begin{array}{r} 200_{\text{three}} \\ \times\ 2_{\text{three}} \\ \hline \end{array}$ (c) $\begin{array}{r} 212_{\text{three}} \\ \times\ 21_{\text{three}} \\ \hline \end{array}$

11. Check Exercises 9(a) and 10(a) using base ten numerals.

12. (a) What is $10_{five} \times 10_{five}$? $10_{five} \times 100_{five}$?

(b) What is $10_{seven} \times 10_{seven}$? $10_{seven} \times 100_{seven}$?

(c) What is $10_{two} \times 10_{two}$? $10_{two} \times 100_{two}$?

(d) What is the pattern you observe in (a), (b), and (c)?

13. Compute the following, regrouping as necessary:

(a) $\begin{array}{r} 4 \text{ yds. } 2 \text{ ft. } 5 \text{ in.} \\ \underline{\times\ 3} \end{array}$ (b) $\begin{array}{r} 2 \text{ m. } 27 \text{ cm. } 4 \text{ mm.} \\ \underline{\times\ 5} \end{array}$

3.4 Division

Teachers may wish to regard the topic of division as optional for some classes, or to treat it along with the study of multiplication. Both discussion and exercises are included here for those who wish to pursue the section. Once again, teachers are urged not to spend an excessive amount of time on this work with other number bases in their seventh grade classes.

For counting numbers we may think of multiplication as repeated addition. As division is to undo multiplication, we may think of division as repeated subtraction. The following example shows continued subtraction of 5 from 20. We may indicate this process by writing $20 \div 5 = 4$, where 4 represents the number of subtractions possible before reaching a remainder less than 5.

$$\begin{array}{r} 20 \\ \underline{-5} \\ 15 \\ \underline{-5} \\ 10 \\ \underline{-5} \\ 5 \\ \underline{-5} \\ 0 \end{array}$$

This approach to division may be shown with a set of 20 objects partitioned into subsets of fives. The model shows 4 subsets, each containing 5 members, thus illustrating $20 \div 5 = 4$.

Perhaps the most useful technique for teaching division relates division to multiplication. As examples, in finding the missing factors in multiplication problems such as

$$4 \times ? = 20 \qquad \text{and} \qquad 5 \times ? = 20,$$

we are actually finding the missing quotients in the corresponding division problems,

$$20 \div 4 = ? \qquad \text{and} \qquad 20 \div 5 = ?$$

Students should be well drilled in the relationship between these two operations. Not only will this help them to see how the multiplication tables in various bases can be used to do division problems, but it will also give them good background for their work in algebra.

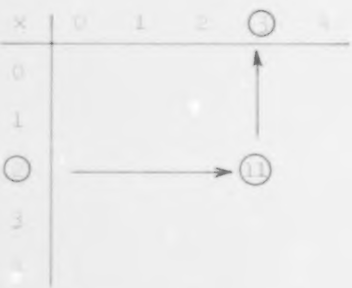
Consider the following division problems using base five numerals:

$$11_{\text{five}} \div 2_{\text{five}} = ?_{\text{five}}$$

The corresponding multiplication problem can be written as:

$$2_{\text{five}} \times ?_{\text{five}} = 11_{\text{five}}$$

2_{five} times what number gives 11_{five}?



The missing factor, 3_{five}, can be found in the base five multiplication table as shown. Thus, we have the quotient:

$$11_{\text{five}} \div 2_{\text{five}} = 3_{\text{five}}.$$

Other simple divisions using base five numerals can be performed in a similar way using the base five multiplication table. Use this method to verify each of the following:

$$14_{\text{five}} \div 3_{\text{five}} = 3_{\text{five}}$$

$$22_{\text{five}} \div 4_{\text{five}} = 3_{\text{five}}$$

$$13_{\text{five}} \div 2_{\text{five}} = 4_{\text{five}}.$$

While the use of physical models is a good learning device, we cannot depend upon drawing models to represent divisions where larger numbers are concerned. Nor can we always find quotients by direct inspection of the multiplication table. Consequently using an algorithm, which is a way of recording or of processing one's thinking, is helpful.

Think of computing $760 \div 20$ with decimal numerals using two forms of the algorithm shown.

$$\begin{array}{r|l} & \underline{\;38} \\ & \;\;\,8 \\ & \;30 \\ \hline 20 & 760 \\ & \underline{600} \\ & 160 \\ & \underline{160} \\ & \;\;\;0 \end{array} \qquad \text{or} \qquad \begin{array}{r|r|l} \cline{2-2} 20 & 760 & \\ & 600 & 30 \\ \cline{2-3} & 160 & \\ & 160 & 8 \\ \cline{2-3} & 0 & 38 \end{array}$$

Either form of the division records the thinking required to answer the division $760 \div 20 = ?$ which is suggested by the sentence $20 \times ? = 760$.

In terms of partitioning a set, we may consider that:

(1) a set of 760 objects has been partitioned into 20 equivalent subsets, each containing 38 members; or

(2) a set of 760 objects has been partitioned into subsets containing 20 members each, with a total of 38 such subsets.

To use such an algorithm with base five numerals we need to recall the role that the various groupings such as 10_{five}, 100_{five}, and 1000_{five} play as factors in multiplications. For example:

$$23_{\text{five}} \times 10_{\text{five}} = 230_{\text{five}}$$

$$23_{\text{five}} \times 100_{\text{five}} = 2300_{\text{five}}$$

$$23_{\text{five}} \times 1000_{\text{five}} = 23000_{\text{five}}.$$

We can use these facts to do certain divisions such as:

$$230_{\text{five}} \div 10_{\text{five}} = 23_{\text{five}}$$

$$2300_{\text{five}} \div 100_{\text{five}} = 23_{\text{five}}$$

$$23000_{\text{five}} \div 23_{\text{five}} = 1000_{\text{five}}.$$

How do we approach a division such as $233_{\text{five}} \div 4_{\text{five}}$? Let us use the division algorithm and certain base five multiplication combinations.

$$\begin{array}{r|l|l} \cline{2-3} 4_{\text{five}} & 233_{\text{five}} & \\ & \underline{220} & 30_{\text{five}} \text{ ——— } (4_{\text{five}} \times 30_{\text{five}} = 220_{\text{five}}) \\ & \;13 & \\ & \underline{\;13} & \;\;2_{\text{five}} \text{ ——— } (4_{\text{five}} \times 2_{\text{five}} = 13_{\text{five}}) \\ \cline{2-3} & \;\;0 & 32_{\text{five}} \end{array}$$

Note that the products listed at the right in the illustration show the corresponding multiplications needed. Each, of course, comes from a basic multiplication found in the table.

We can apply a similar procedure for divisions with two-digit divisors.

$$
\begin{array}{r|r}
23_{\text{five}}\;\overline{)\,32141_{\text{five}}} & \\
23000 & 1000 \\
\hline
4141 & \\
2300 & 100 \\
\hline
1341 & \\
1240 & 30 \\
\hline
101 & \\
101 & 2 \\
\hline
0 & 1132_{\text{five}}
\end{array}
\qquad\qquad
\begin{array}{r|r}
32_{\text{five}}\;\overline{)\,40243_{\text{five}}} & \\
32000 & 1000 \\
\hline
3243 & \\
3200 & 100 \\
\hline
43 & \\
32 & 1 \\
\hline
11_{\text{five}} & 1101_{\text{five}} \\
\text{(remainder)} &
\end{array}
$$

These two divisions may be expressed simply as:

$$32141_{\text{five}} \div 23_{\text{five}} = 1132_{\text{five}}$$

and

$$40243_{\text{five}} \div 32_{\text{five}} = 1101_{\text{five}} \text{ with remainder } 11_{\text{five}}.$$

These divisions can be checked using base ten numerals. Teachers may prefer to check division problems by using multiplication in the indicated base. The two division examples in base five are checked by this method.

$$
\begin{array}{r}
1132_{\text{five}} \\
\times\;\;23_{\text{five}} \\
\hline
4001 \\
2314 \\
\hline
32141_{\text{five}}
\end{array}
\qquad\qquad
\begin{array}{r}
1101_{\text{five}} \\
\times\;\;32_{\text{five}} \\
\hline
2202 \\
33030 \\
\hline
40232_{\text{five}} \\
+\;11_{\text{five}} \\
\hline
40243_{\text{five}}
\end{array}
\;\text{(remainder)}
$$

It is suggested that division in any base other than ten be approached by constructing the table of basic multiplication combinations for that base. Such an approach strengthens the concept that division is the inverse of multiplication. To have meaning, division in any base must relate to multiplication in that base.

Class Exercises

14. Perform the indicated divisions.

(a) $112_{five} \div 4_{five}$ (c) $302_{five} \div 12_{five}$

(b) $1031_{five} \div 3_{five}$ (d) $1040_{five} \div 23_{five}$

15. Set up a table of multiplication combinations for base three. Perform the indicated divisions and check by multiplication.

(a) $121_{three} \div 2_{three}$ (c) $10010_{three} \div 20_{three}$

(b) $1012_{three} \div 22_{three}$ (d) $1220_{three} \div 12_{three}$

Chapter Exercises

1. Use the array you made in Class Exercise 1 and add each of the following. Check your answers using base ten.

(a) $$\begin{array}{r} 235_{eight} \\ +\ 175_{eight} \\ \hline \end{array}$$ (c) $$\begin{array}{r} 307_{eight} \\ +\ 175_{eight} \\ \hline \end{array}$$ (e) $$\begin{array}{r} 44_{eight} \\ +\ 237_{eight} \\ \hline \end{array}$$

(b) $$\begin{array}{r} 1064_{eight} \\ +\ 253_{eight} \\ \hline \end{array}$$ (d) $$\begin{array}{r} 121_{eight} \\ +\ 567_{eight} \\ \hline \end{array}$$

2. Subtract in base eight, using the numerals in parts (a), (b), and (c) of Exercise 1. Check your subtraction using base ten numerals.

3. Multiply using base eight numerals:

(a) $$\begin{array}{r} 523_{eight} \\ \times\ \ 5_{eight} \\ \hline \end{array}$$ (b) $$\begin{array}{r} 741_{eight} \\ \times\ 36_{eight} \\ \hline \end{array}$$

4. Divide using base eight numerals:

(a) $4_{eight}\overline{)34_{eight}}$ (b) $13_{eight}\overline{)5146_{eight}}$

5. Write a division sentence suggested by each of the following products:

(a) Base ten:

$9 \times 8 = 72$

$2 \times 40 = 80$

$5 \times n = 25$

$4 \times n = 24$

$n \times 10 = 100$

(b) Base five:

$2_{five} \times 3_{five} = 11_{five}$

[illegible]$_{five} \times n_{five} = 31_{five}$

(c) Base seven:

$4_{seven} \times 5_{seven} = 26_{seven}$

$4_{seven} \times n_{seven} = 33_{seven}$

[illegible]$_{seven} \times n_{seven} = 100_{seven}$

6. Find the value of each n (noting the base indicated) in Exercise 5(c).

7. Some completed problems are given below; name the base in which each problem is stated.

(a)
$$\begin{array}{r} 32 \\ +\ 14 \\ \hline 101 \end{array}$$

(c)
$$\begin{array}{r} 24 \\ \times\ 4 \\ \hline 132 \end{array}$$

(e)
$$\begin{array}{r} 13 \\ \times\ 1 \\ \hline 13 \end{array}$$

(b)
$$\begin{array}{r} 65 \\ \text{[illegible]} \\ +\ 61 \\ \hline \text{[illegible]} \end{array}$$

(d)
$$\begin{array}{r} 33 \\ \times\ 14 \\ \hline 242 \\ 33 \\ \hline 1122 \end{array}$$

(f)
$$\begin{array}{r} 24 \\ +\ 24 \\ \hline 48 \end{array}$$

8. In base ten the final digit of an even number is 0, 2, 4, 6, 8.

(a) For base two numeration, what is the final symbol of an even number?

(b) Answer the same question for base three.

9. In base ten numeration the final digit of the square of a number is 0, 1, 4, 5, 6, or 9.

(a) In base two notation what may be said about the final symbol of the square of a number?

(b) Answer the same question for base three.

10. Is it possible to substitute base ten digits for the letters in

$$\begin{array}{r} \text{SEVEN} \\ +\ \text{EIGHT} \\ \hline \text{ELEVEN} \end{array}$$

so that a correct addition results? A letter always represents the same digit and no digit is represented by more than one letter.

11. Complete the following addition and multiplication problems in base seven:

(a)
$$\begin{array}{r} 204_{\text{seven}} \\ 354_{\text{seven}} \\ +\ ???_{\text{seven}} \\ \hline 1116_{\text{seven}} \end{array}$$

(b)
$$\begin{array}{r} 314_{\text{seven}} \\ \times\ ?_{\text{seven}} \\ \hline 3105_{\text{seven}} \end{array}$$

(c)
$$\begin{array}{r} ???_{\text{seven}} \\ \times\ 54_{\text{seven}} \\ \hline 36401_{\text{seven}} \end{array}$$

Answers to Class Exercises

1.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	[illegible]	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

2. (a) 40_{five} (b) 112_{five} (c) 400_{eight} (d) 616_{eight}

3. (a)
$$\begin{array}{rl} 22_{five} & = (2 \times 5) + (2 \times 1) = 12 \\ + \; 13_{five} & = (1 \times 5) + (3 \times 1) = \underline{\;8} \\ \hline 40_{five} & = (4 \times 5) + (0 \times 1) = 20 \end{array}$$

(b)
$$\begin{array}{rl} 43_{five} & = (4 \times 5) + (3 \times 1) \qquad\qquad = 23 \\ + \; 14_{five} & = (1 \times 5) + (4 \times 1) \qquad\qquad = \underline{\;9} \\ \hline 112_{five} & = (1 \times 5^2) + (1 \times 5) + (2 \times 1) = 32 \end{array}$$

(c)
$$\begin{array}{rl} 177_{eight} & = (1 \times 8^2) + (7 \times 8) + (7 \times 1) = 127 \\ + \; 201_{eight} & = (2 \times 8^2) + (0 \times 8) + (1 \times 1) = \underline{129} \\ \hline 400_{eight} & = (4 \times 8^2) + (0 \times 8) + (0 \times 1) = 256 \end{array}$$

(d)
$$\begin{array}{rl} 521_{eight} & = (5 \times 8^2) + (2 \times 8) + (1 \times 1) = 337 \\ + \; 75_{eight} & = (0 \times 8^2) + (7 \times 8) + (5 \times 1) = \underline{\;61} \\ \hline 616_{eight} & = (6 \times 8^2) + (1 \times 8) + (6 \times 1) = 398 \end{array}$$

4. (a)
$$\begin{array}{rcl} 75_{ten} & = & 300_{five} \\ + \; 243_{ten} & = & 1433_{five} \\ \hline 318_{ten} & = & 2233_{five} \end{array}$$

(b)
$$\begin{array}{rcl} 25_{ten} & = & 100_{five} \\ + \; 104_{ten} & = & 404_{five} \\ \hline 129_{ten} & = & 1004_{five} \end{array}$$

5. (a) 240_{ten} (b) 100_{five} (c) 30_{five} (d) 340 (e) 200_{seven}

6. (a) 26_{twelve} (d) 14_{five}

(b) 101_{three} (e) 7_{eight}

(c) 26_{seven} (f) 11_{two}

8. (a) > (b) = (c) < (d) >

9. (a) 112_{five} (b) 23_{five} (c) 13404_{five}

10. (a) 2100_{seven} (b) 1100_{three} (c) 12222_{three}

12. (a) 100_{five}, 1000_{five} (c) 100_{two}, 1000_{two}

(b) 100_{seven}, 1000_{seven} (d) Regardless of the base used, $10 \times 10 = 100$ and $10 \times 100 = 1000$.

13. (a) 14 yd. 1 ft. 5 in. (b) 11 m. 37 cm.

14. (a) 13_{five} (c) 21_{five}

(b) 142_{five} (d) 21_{five} + remainder [illegible]$_{\text{five}}$

15.

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

(a) 22_{three}

(b) 11_{three}

(c) 112_{three}

(d) 101_{three}, (remainder 1_{three})

Chapter 4

MATHEMATICAL SYSTEMS

Introduction

Throughout the work of grades 7, 8, and 9 we are concerned with the development of various sets of numbers together with operations on these numbers. It is just as important that junior high school youngsters see the structure of these systems as that they be able to manipulate the elements of any specific system under some given operation.

In chapters to follow we shall carefully explore various number systems and their properties as they may be developed in the junior high school. However, in this chapter we shall explore several abstract systems in order to illustrate and name some important properties of numbers. This is not the manner in which we recommend that these properties be developed for all seventh grade youngsters! For them we suggest an introduction via a more concrete and familiar situation such as will be explored in the ensuing chapters. The abstract development described here could then well follow later in the year.

Youngsters enjoy working with abstractions, especially after they have completed the mathematics program of grades K-6. Often this helps them to see mathematics in a new light and to see the structure in what may have previously been a jumble of unrelated mechanical procedures.

It is also true, however, that seventh graders may develop the habit of asking why this material need be studied. "Why do I have to know the commutative property?" they will ask. As these various properties are developed in this chapter, examples of their applications are also occasionally given. Such a procedure may well be followed as you present this material to your students. However, you may also have to admit periodically that the "pay-off" for these properties will not come until a later date. The distributive property, for example, is very useful in the study of elementary algebra. Much like a good mystery novel, we are developing the essential features of the plot at this time, but must wait until a later date to unravel the rest of the story.

4.1 Binary Operation

> Given a set of elements, a *binary operation* is a rule whereby to each pair of elements of the set there corresponds exactly one element.

Seventh graders will be familiar with the concept of a binary operation from their work in arithmetic, although perhaps not with the language. Each of the fundamental operations of arithmetic--addition, subtraction, multiplication, and division--is a binary operation. For example, addition is a binary operation in that this operation assigns exactly one number to any two given numbers. Thus, given 7 and 8, we have $7 + 8 = 15$. Similarly, multiplication is a binary operation; given the numbers 7 and 8, we have $7 \times 8 = 56$.

The term binary is used to emphasize the fact that such operations are only done with two elements at a time. Even in addition we do not add three numbers at once; we add two of them and then to that sum we add the third.

Is subtraction a binary operation? Although $7 - 3 = 4$, we should note that the student who has had no experience with negative numbers is not able to find a number to correspond to $3 - 7$. However, there is a definite number that corresponds to $3 - 7$ and we do, therefore, consider subtraction to be a binary operation. Some students will recognize that $3 - 7 = {}^-4$, whereas others will learn this fact in a later course.

(It is interesting to note that not all mathematicians will agree on this point. Some will argue that the operation is not a binary one unless it produces a result which is itself a member of the original set of elements.)

An interesting way to introduce youngsters to this concept of an operation is to have them "discover" the meaning of certain abstract operations. Consider, for example, a binary operation symbolized by * . The symbol * is a sign of the operation. Thus, $2 * 3$ tells you to operate on 2 and 3 in a certain way. Following are several illustrations of the use of this operation. See if you can discover the meaning of *.

$2 * 3 = 6$ $\qquad$ $4 * 8 = 13$

$3 * 2 = 6$ $\qquad$ $5 * 0 = 6$

In this case the binary operation * tells you to add one to the sum of the two given numbers. That is, for any numbers a and b:

$$a * b = (a + b) + 1.$$

Note that you also obtain the same result by defining the operation * in either of the following ways:

$$a * b = (a + 1) + b$$
$$a * b = a + (b + 1)$$

An interesting classroom activity is to have youngsters invent their own binary operation, present examples of its use, and allow other members of the class to discover its meaning. You will need to set up some sort of guidelines here or this activity can soon get out of hand. For example, it will be almost impossible for a class to discover the meaning of a binary operation that means you are to increase the first number by 5, decrease the second number by 3, and then find their product! Be sure to keep the meaning of the operation within reason. The class exercises below are examples of this type of activity.

Class Exercises

Use the examples given to discover the meaning of the operations $\bigcirc$, $\perp$, $\triangle$, $\sim$, $\sqcup$.

1. $[illegible] \bigcirc 5 = 8$
$3 \bigcirc 3 = 5$
$[illegible] \bigcirc [illegible] = 10$
$1 \bigcirc [illegible] = [illegible]$

2. $9 \perp 2 = 8$
$6 \perp 3 = 7$
$8 \perp 10 = 5$
$4 \perp 1[illegible] = 8$

3. $3 \triangle 5 = 5$
$7 \triangle 1 = 7$
$6 \triangle 7 = 0$
$8 \triangle 3 = 9$

4. $5 \sim 5 = 10$
$4 \sim 3 = 5$
$5 \sim 0 = 6$
$5 \sim 1 = 11$

5. $3 \sqcup 4 = 5$
$1 \sqcup 2 = 5$
$2 \sqcup 8 = 3$
$7 \sqcup 5 = 0$

[illegible] A Mathematical System

A mathematical system is a set of elements with one or more binary operations defined on the set. The elements do not have to be numbers, although they most often are as they are encountered in a seventh grade mathematics class. It may be interesting first to explore the properties of an abstract system where the elements are not numbers.

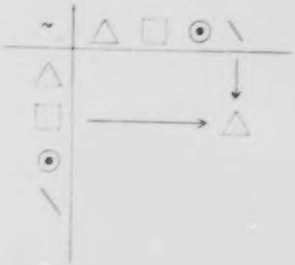
Let us consider a set, M, of elements:

$$M = \{\triangle, \square, \odot, \diagdown\}.$$

Also consider a binary operation, ~, that combines any two members of set M. We can define this operation by means of the following table:

~	△	□	⊙	\
△	△	□	⊙	\
□	□	⊙	\	△
⊙	⊙	\	△	□
\	\	△	□	⊙

This table is read by locating the first of two given elements in the row headings to the left of the table. The second element is then located in the column headings at the top of the table. The result is found within the table where the row and column intersect. For example, to find □ ~ \ , we first look



in the row headings at the left of the table until we find the first symbol, □, and then move to the right to the column headed by the second element, \ , to find the result, △. Thus, □ ~ \ = △. In a similar manner, verify that each of the following is correct:

\ ~ ⊙ = □
△ ~ ⊙ = ⊙
⊙ ~ □ = \

We now have a mathematical system consisting of the set M and the binary operation ~. Note that it really does not matter what the operation ~ means; the operation is defined by the table and we learn about it by comparing the table to discover the properties. Here it is worthwhile to let students examine the table and attempt to discover some properties on their own. What can they discover? What can you discover?

For one thing, all of the entries in the table are members of set M; no new symbol appears. We describe this property by saying that the set M is closed with respect to the operation ~ . This is the *closure property*. In general:

> A set S is said to be *closed* under
> a binary operation * if for any
> elements x and y of S, x * y
> is an element of S.

Thus, the set of whole numbers is closed under addition because the sum of any two whole numbers is a whole number. Do you see that the set of whole numbers is *not* closed under subtraction?

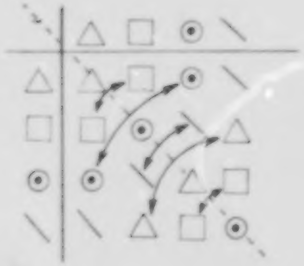
Let us see what else we can discover from the table given at the beginning of this section. Compare your answers to parts (a) and (b) of each of the following examples.

1. (a) □ ~ ⊙ = ?   2. (a) \ ~ □ = ?   3. (a) △ ~ ⊙ = ?
   (b) ⊙ ~ □ = ?   (b) □ ~ \ = ?   (b) ⊙ ~ △ = ?

Do you see that the answers are the same in each pair? Indeed this will be true for any pair of elements selected from this table, as you can verify by examination. We say that the operation ~ is *commutative*. By this we mean that the result is independent of the *order* in which the operation is performed. Not all operations are commutative. In general:

> An operation * defined on a set S
> is said to be *commutative* if for any
> elements x and y of S:
>
> x * y = y * x.

There is an easy way to discover whether or not an operation is commutative if one has a table that defines the operations. If there is symmetry with respect to a diagonal line drawn from the upper left to the lower right corner, then we have commutativity.



Note that each element on one side of the dotted line is symmetric to and corresponds to a like element of the other side of the line. Results that come from combining two elements in different order always occur in these corresponding positions in the table. Hence, if reordering the elements operated on does not change the result, these corresponding entries must always agree.

Of course, it is not always feasible nor possible to construct a table that defines an operation. For example, if our original set of elements were to consist of the set of real numbers, we would have an infinite collection that could not be accommodated in a table. In such a case the diagonal line test for commutativity could not be used.

Let's consider one more property in this section, this time involving three elements. Since a binary operation relates only two members of a set, we need to make use of parentheses in order to determine which pair of elements to combine first. Without parentheses, the operation might be ambiguous. For example, consider the problem

$$12 \div 6 \div 2.$$

If one divides from left to right the result is $(12 \div 6) = 2$ and $2 \div 2 = 1$. On the other hand, if one divides 6 by 2 first, the result is $12 \div (6 \div 2) = 12 \div 3 = 4$. Thus, the problem, as originally stated, is ambiguous unless parentheses are used or some agreement is made concerning the order in which the elements are grouped for the binary operations. On the other hand, a statement such as $12 + 6 + 2$ is not ambiguous since $(12 + 6) + 2 = 18 + 2 = 20$ and also $12 + (6 + 2) = 12 + 8 = 20$. Here the grouping does not affect the result. Of course, addition still remains a binary operation; only two elements are added at a time. The point is that in addition, the way the elements are grouped does not affect the result, whereas in division it does.

Now let us evaluate an expression involving three elements of set M keeping in mind that operations within parentheses are to be done first.

$$\begin{aligned}(\odot \sim \square) \sim \triangle &= \diagdown \sim \triangle \\ &= \diagdown\end{aligned}$$

Here the same three elements are grouped in a different way:

$$\begin{aligned}\odot \sim (\square \sim \triangle) &= \odot \sim \square \\ &= \diagdown\end{aligned}$$

Note that the result is the same in each case. That is,

$$(\odot \sim \square) \sim \triangle = \odot \sim (\square \sim \triangle).$$

Will this be true for all arrangements of three elements from set M ? Evaluate the following:

1. (a) $(\diagdown \sim \odot) \sim \square = ?$
   (b) $\diagdown \sim (\odot \sim \square) = ?$
2. (a) $(\square \sim \diagdown) \sim \odot = ?$
   (b) $\square \sim (\diagdown \sim \odot) = ?$

You should find that the answers for each pair of examples are the same and this will be true for any similar arrangement of three elements from set M. We say that the operation $\sim$ is associative. In general:

> An operation $*$ defined on a set S is said to be associative if for any elements x, y, and z of S:
>
> $$(x * y) * z = x * (y * z).$$

You should note that it is not possible to tell whether an operation is associative by looking at a table. Nor may one assume associativity on the basis of several examples only. Actually every combination of three elements would have to be tried in order to prove associativity. On the other hand, if just one example can be found when the property does not hold, then the operation is not associative. Likewise, if the mathematical system is not closed, then the associative property cannot hold.

Class Exercises

Use the following information for Exercises 6-13:

$A = \{a, b, c\}$

$\otimes$	a	b	c
a	b	c	a
b	c	b	a
c	a	c	b

$L = \{1, 3, 5, 7\}$

$\oplus$	1	3	5	7
1	3	7	1	5
3	5	7	1	3
5	3	1	5	7
7	7	5	3	1

6. $a \otimes c = ?$

7. $3 \oplus 7 = ?$

8. $a \otimes (b \otimes c) = ?$

9. $(7 \oplus 3) \oplus 1 = ?$

10. Is the set A closed with respect to $\otimes$?

11. Is the set L closed with respect to $\oplus$?

12. Does $3 \oplus 5 = 5 \oplus 3$? Is the operation $\oplus$ a commutative one? (Try $7 \oplus 3$ and $3 \oplus 7$.)

13. Is the operation $\otimes$ a commutative operation?

For each of the following described sets and operations determine whether the set is closed. Find which operations are commutative and which are associative.

14. Set: All counting numbers between 25 and 75.
    Operation: Choose the smaller number.
    Example: 28 combined with 36 produces 28.

15. Set: All even numbers between 39 and 61.
    Operation: Choose the first number.
    Example: 52 combined with 46 produces 52;
    46 combined with 52 produces 46.

4.3 Mathematical Systems - Additional Properties

Let us return to set M of the previous section and discover several additional properties of the mathematical system developed there. For convenience, here again is the table defining the operation ~ :

	△	□	⊙	╲
△	△	□	⊙	╲
□	□	⊙	╲	△
⊙	⊙	╲	△	□
╲	╲	△	□	⊙

Now note the following:

$$\square \sim \triangle = \square \qquad \triangle \sim \square = \square$$
$$\odot \sim \triangle = \odot \qquad \triangle \sim \odot = \odot$$
$$\diagdown \sim \triangle = \diagdown \qquad \triangle \sim \diagdown = \diagdown$$
$$\triangle \sim \triangle = \triangle$$

Do you see that the combination of any element of M with △ produces the original member of M ? In other words, the element △ plays the same role here as 0 plays in addition. Recall that the sum of any number and zero is that number. Thus, for any number n,

$$n + 0 = 0 + n = n.$$

The number 1 plays a corresponding role with respect to multiplication. For any number n,

$$n \cdot 1 = 1 \cdot n = n.$$

Likewise, the element △ plays the same role with the operation ~ .

$$n \sim \triangle = \triangle \sim n = n.$$

We call such elements *identity elements*. An identity element does not change the identity of any element with which it is combined through the operation.

In general:

> An element I is said to be an identity element for the operation $*$, defined on a set S, if $x * I = I * x = x$ for each element x of S.

Note that 0 is the identity element for addition, and that 1 is the identity element for multiplication. Is there an identity element for subtraction or for division? Explain your answer.

As another example consider the following table for an operation which we might call ★. First confirm that the operation ★ is a commutative one.

★	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Is there an identity element for ★ ? Could it be A ? If so, then how must the element A behave with respect to the operation ★ ? Is B ★ A = A ★ B = B ? We see, from the table, that the answer to the question is "no" and thus, A cannot be an identity element. Neither can B be an identity element since A ★ B is not A. However, D is an identity for ★, since

A ★ D = D ★ A = A,
B ★ D = D ★ B = B,
C ★ D = D ★ C = C,
D ★ D = D.

In the table compare the column under D with the column under the ★. Compare the row to the right of D with the row to the right of the ★. What do you notice? Does this suggest a way to look for an identity element when you are given a table for the operation?

Whenever we have an identity element for an operation, it may be that we also have what are called inverse elements. When the operation is multiplication for real numbers, the identity element is 1. If the product of two numbers, a and b, is the multiplicative identity, 1, then we call each of the numbers the multiplicative inverse of the other. Thus, the multiplicative inverse of 3 is $\frac{1}{3}$ since $3 \times \frac{1}{3} = 1$. You may recognize the multiplicative inverse of a number as its reciprocal.

Suppose the operation is addition. Here 0 is the identity element and we call two numbers additive inverses if their sum is 0; that is, combining the two numbers by addition gives 0. For example, the additive inverse of 3 is $^-3$ since $3 + (^-3) = 0$. We say the additive inverse of a number is its opposite.

Let us return to the set M described earlier in this section. Recall that we found that the set contained an identity element, namely △. Now let us see if each element has an inverse with respect to the operation ~ . For example, to determine the inverse of □ we must find some element to combine with □ that will produce the identity △. We find this to be ╲ since □ ~ ╲ = △. Similarly, the inverse of ╲ is □ since ╲ ~ □ = △. The inverse of △ is △ and the inverse of ⊙ is ⊙:

△ ~ △ = △

⊙ ~ ⊙ = △

In general:

> Two elements x and y of set S
> are said to be inverses of each
> other under a binary operation * if
>
> $$x * y = y * x = I$$
>
> where I is the identity element for
> the given set S.

It is possible that within a given set only certain elements have inverses. However, it is impossible for any elements to have inverses if there is no identity element.

The concepts of identity and inverse elements are important ones in the development of mathematics. Let's pause to consider several illustrations of their use in elementary mathematics. Additional illustrations will be given in later chapters.

The identity element for multiplication, 1, is useful in explaining the principle involved in simplifying fractions. For example:

$$\frac{9}{12} = \frac{3}{4} \times \frac{3}{3}$$
$$= \frac{3}{4} \times 1$$
$$= \frac{3}{4}$$

A similar procedure is used in simplifying algebraic fractions;

$$\frac{2x - 4}{3x - 6} = \frac{2(x-2)}{3(x-2)} \qquad \text{by factoring } (x \neq 2)$$

$$= \frac{2}{3} \cdot \frac{x-2}{x-2}$$

$$= \frac{2}{3} \cdot 1$$

$$= \frac{2}{3}$$

We make use of both of the concepts developed in this section in solving simple equations. Below is a detailed exploration of the solution for the equation $2x + 3 = 7$. (Of course, we normally do not go through each of these steps in this formal a manner.) Note the use of identity elements and inverses of elements in this development.

$$2x + 3 = 7$$

$$(2x + 3) + ({}^-3) = 7 + ({}^-3) \qquad \text{(The additive inverse of } 3 \text{ is } {}^-3\text{)}$$

$$2x + (3 + {}^-3) = 7 + ({}^-3) \qquad \text{(By the associative property for addition)}$$

$$2x + 0 = 4 \qquad \text{(Here } 0 \text{ is the additive identity)}$$

$$2x = 4$$

$$\frac{1}{2}(2x) = \frac{1}{2}(4) \qquad \text{(The multiplicative inverse of } 2 \text{ is } \tfrac{1}{2}\text{)}$$

$$\left(\frac{1}{2} \cdot 2\right) \cdot x = \frac{1}{2}(4) \qquad \text{(By the associative property for multiplication)}$$

$$1 \cdot x = 2 \qquad \text{(Here } 1 \text{ is the multiplicative identity)}$$

Thus: $x = 2$

Class Exercises

Use the accompanying table to answer the following questions relative to set $K = \{a,b,c,d,e\}$ and operation $*$ given in the table.

*	a	b	c	d	e
a	d	e	a	b	c
b	e	a	b	c	d
c	a	b	c	d	e
d	b	c	d	e	a
e	c	d	e	a	b

16. $b * c = ?$

17. $(a * a) * d = ?$

18. $a * (d * e) = ?$

19. Is the set K closed with respect to the operation $*$?

20. Is $*$ a commutative operation?

21. Does the set K contain an identity element with respect to $*$?
    If so, what is it?

22. Name the inverse of each of the elements of set K.

4.4 Clock Arithmetic

The numerous properties that we have explored thus far are all important ones with which seventh graders must become familiar. Of course, they should become acquainted with the concepts, if not the actual language, long before they ever enter the seventh grade.

It is frequently difficult to convince junior high school youngsters of the importance of these concepts if they are explained only in terms of ordinary arithmetic. They see little to get excited about in the fact that $2 + 3 = 3 + 2$, or that $5 + 0 = 5$. It is for this reason that it is often advisable to present these ideas in unique settings where possible. Of course, this depends upon the background and ability of the group in question.

One interesting mathematical system that junior high school youngsters enjoy exploring is clock arithmetic. On our twelve-hour clock it is perfectly reasonable to say that $8 + 7 = 3$. (That is, seven hours after 8 o'clock it will be 3 o'clock.) Now we can set up an entire mathematical system based on addition on the twelve-hour clock. Youngsters can be encouraged to complete an addition table for this system and to explore its properties.

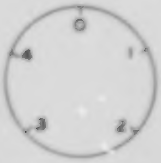
Similar activities can be developed around the twenty-four hour clock used in the navy where $9 + 11 = 20$ and $23 + 5 = 4$.

Class Exercises

Consider the elements involved in a twelve-hour clock together with the operation of addition.

23. $9 + 10 = ?$
24. $11 + 11 = ?$
25. Does this system have an identity element? If so, what is it?
26. What is the inverse of 5 with respect to addition? Explain your answer.

Let us finally turn our attention to arithmetic on a clock with fewer positions. Consider one with a set of five symbols: {0,1,2,3,4}.



We may introduce a binary operation on these symbols by considering clockwise rotations. This definition will be in terms of adding or combining one clockwise rotation with another clockwise rotation. For this reason we call the operation addition and use the symbol +.

We shall consider the starting point to be the position labeled 0. (This is standard practice, although we could have used 3 or any other symbol as well.) For example, 3 + 4 means that we start at 0 and move to position 3. From this position we move 4 more steps to arrive at position 2. Thus, 3 + 4 produces 2. We might simply write this as 3 + 4 = 2. However, this type of arithmetic is an example of modular arithmetic where such an addition is usually written

$$3 + 4 \equiv 2 (\text{mod } 5)$$

and is read:

"Three plus four is equivalent to two, modulo 5."

The word "mod" stands for modulus or modulo.

It is customary in the theory of numbers to treat two numbers as equivalent in a given modulus if they have the same remainders when divided by the modulus. For example, $7 \equiv 2 (\text{mod } 5)$ since both 7 and 2 give a remainder of 2 when divided by 5. However, in the application of this idea to our clock arithmetic, we restrict ourselves only to the elements in the finite set {0,1,2,3,4}. Examples and discussions of this application may be found in many Junior High books. This approach through clock arithmetic is a simple way to start such discussions, and, of course, provides opportunities to examine again some of the basic properties we are developing.

In this example the modulus is 5 which means that there are five positions on the face of the clock. The symbol ≡ indicates that 3 + 4 and 2 are equivalent on the clock. Verify that each of the following is correct:

$$2 + 3 \equiv 0 \ (\text{mod } 5)$$
$$4 + 4 \equiv 3 \ (\text{mod } 5)$$
$$3 + 3 \equiv 1 \ (\text{mod } 5)$$

A second operation may also be introduced on the symbols 0, 1, 2, 3, 4. Since this operation is related in a familiar way to the operation that we have called addition, this new binary operation is called multiplication and the familiar × symbol is used.

By multiplication on this clock we shall mean repeated clockwise rotations. Thus, 2 × 4 means 4 + 4 and 4 × 3 means 3 + 3 + 3 + 3. Verify that each of the following is correct:

$$2 \times 4 \equiv 3 \pmod 5$$
$$4 \times 3 \equiv 2 \pmod 5$$
$$3 \times 3 \equiv 4 \pmod 5.$$

Here are the completed tables for addition and multiplication in this system. Seventh graders should be encouraged to complete and use--but not memorize--such tables. In a seventh grade class you would normally develop these tables rather than present them in completed form as is done here.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

We can once again explore each of these systems for the various properties discussed earlier. There is one additional property that we need to explore that includes two operations. Consider these problems:

(a) 3 × (4 + 2) =
3 × 1 =
3

(b) (3 × 4) + (3 × 2) =
2 + 1 =
3

Remember, we operate within the parentheses first. Note that we obtain the same result either way. Is this a pattern or an accident? Do we always get the same answer if we add before we multiply as we do when we multiply first and then add?

Try each of the following pairs of problems:

(a) 2 × (3 + 4) and (2 × 3) + (2 × 4)
(b) 4 × (3 + 2) and (4 × 3) + (4 × 2)

The answers are the same for each pair and will be for all possible similar combinations of three elements. We say that multiplication distributes over addition. In short we call this the distributive property, a property that relates the two operations of addition and multiplication.

> Given a set M and two binary operations * and o defined on M. The operation * distributes over the operation o if
>
> $x * (y \circ z) = (x * y) \circ (x * z)$
>
> for all elements x, y, and z of set M.

Notice that the definition here is quite general and is not limited to a set of numbers and the operations of addition and multiplication. It is true that most of the applications to be encountered by students involve numbers. However, the definition is more general. In fact, when working with sets, the operations of intersection and union are each distributive over the other. You can verify by means of Venn diagrams that intersection distributes over union and also that union distributes over intersection. This is quite different from the case with numbers where multiplication is distributive with respect to addition but addition is not distributive with respect to multiplication.

The distributive property is a very important one for junior high school youngsters to understand. It forms the basis for the work that they do later in algebra in both multiplying and factoring. For example, the distributive property is the justification for such statements as:

$$3a(b + c) = 3ab + 3ac$$

$$x^2y + xy^2 = xy(x + y)$$

$$(x + 4)(x + 3) = x^2 + 7x + 12 .$$

The distributive property also forms the basis for explaining many of the usual arithmetic algorithms. For example, consider the product 9×37:

$$9 \times 37 = 9(30 + 7) = (9 \times 30) + (9 \times 7)$$

In another form:

$$\begin{array}{r l} 37 & \\ \times\ 9 & \\ \hline 63 & (9 \times 7) \\ 270 & (9 \times 30) \\ \hline 333 & \end{array}$$

Of course, we abbreviate the work, but it is nevertheless based upon this most important property.

Although this chapter has included several examples that indicate the importance of the various properties developed, most youngsters will have to accept this importance on faith at first. Later on these properties are used more extensively to justify what otherwise would appear as mechanical operations.

An interesting item that can be described to seventh graders is a hypothetical computing machine that has room for only one decimal place. Thus, the machine would compute $.8 \times .7$ as $.56$ and round this off to $.6$; and it would compute $.8 \times .4$ as $.3$. This is an example of a non-associative operation. For example, suppose the machine had to compute $.8 \times .7 \times .6$:

$$(.8 \times .7) \times .6 = .6 \times .6 = .4;$$

whereas

$$.8 \times (.7 \times .6) = .8 \times .4 = .3.$$

This helps students see that not all operations obey the various properties listed in this chapter.

Class Exercises

The two tables below describe a mathematical system composed of the set [A,B,C,D] and the two operations * and o.

*	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

o	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	A	C
D	A	D	C	B

27. Do you think * distributes over o ? Try several examples.
28. Do you think o distributes over * ? Try several examples.
29. Find the identity elements for * and o.

4.5 *Conclusion*

The major objective of the work developed in this chapter has been to achieve some appreciation of the nature of a mathematical system. Each of the properties developed is of importance and will be further explored in the forthcoming chapters of this text. Junior high school youngsters need to see these properties as they relate to familiar sets of numbers as well as to abstract systems. In general, they enjoy and have many opportunities for creativity as they explore these abstract systems.

Summary

A *binary operation* is a rule whereby to each pair of elements of a set there corresponds exactly one element.

A *mathematical system* is a set of elements with one or more binary operations defined in the set.

A set is *closed* under a binary operation if every two elements of the set combined by the operation give a result which is an element of the set.

A binary operation is *commutative* if, for every two elements, the result of combining them by the operation is independent of the order. If $*$ is the operation and x and y are the elements, then $x * y = y * x$.

A binary operation is *associative* if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third. If $*$ is the operation and x, y, and z are the three elements, then

$$x * (y * z) = (x * y) * z.$$

An *identity* element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are *inverses* of each other under a binary operation if the result of this operation on the two elements is the identity element for that operation.

A binary operation $*$, *distributes* over the binary operation $\circ$ provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

Chapter Exercises

1. Complete an addition and a multiplication table for a mod 4 arithmetic using the set of elements $M = \{0, 1, 2, 3\}$. (Compare the entries of your table with those in the tables given for Class Exercises 27-29. Let $A = 0$, $B = 1$, $C = 2$, and $D = 3$.)

2. Use the tables from Exercise 1 to complete each of the following:

   (a) $2 + 3$
   (b) $3 + 3$
   (c) 2×3
   (d) 3×3
   (e) $3 \times (2 + 3)$

3. Answer each of the following about the mathematical system of multiplication (mod 4).

   (a) Is the set closed under the operation?
   (b) Is the operation commutative?
   (c) Do you think that the operation is associative?
   (d) Which elements have inverses, and what are the pairs of inverse elements?
   (e) Is it true that if a product is 0 then at least one of the factors is 0 ?

4. Consider the set of elements $\{E, O\}$ and the binary operations $+$, $\times$, defined as follows:

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

   (a) Is either operation commutative?
   (b) Is there an identity element for $\times$ in this set?
   (c) Is $\times$ distributive over $+$? That is, does $a \times (b + c) = (a \times b) + (a \times c)$ for all replacements of E or O for a, b, and c ?
   (d) Is $+$ distributive over $\times$?

For each of the following described sets and operations determine whether the set is closed and find which operations are commutative or associative.

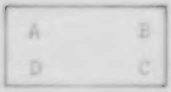
5. Set: All counting numbers less than 50.
   Operation: Multiply the first by 2 and then add the second.
   Example: 3 combined with 5 produces 11 since $2 \cdot 3 + 5 = 11$.

6. Set: All counting numbers.

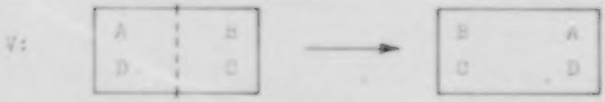
Operation: Raise the first number to a power whose exponent is the second number.

Example: 5 combined with 3 produces 5[illegible].

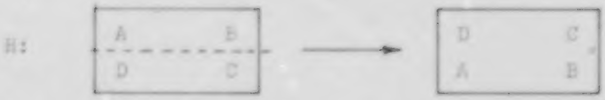
7. Consider a system formed as follows. Place an index card, marked as in the diagram, in "standard position":



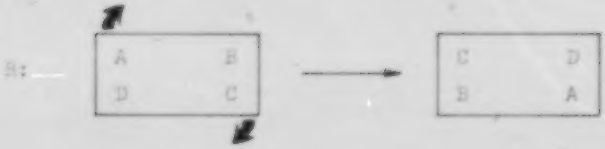
Let us use I to mean leave the card in place. V means to flip the card over using a vertical axis:



H means to flip the card over using a horizontal axis.



R means to rotate the card halfway around in the direction indicated.



Our set of elements is {I,V,H,R}. The operation will be "ANTH" which means to do one thing "and then" do another. Thus, "H ANTH V" means to flip the card over using a horizontal axis, and then flip the card over again using a vertical axis. Try it with an actual card. You should find that H ANTH V = R.

(a) Complete the following table for the operation ANTH. Some entries are already given for you.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Examine the table for the operation ANTH.

(b) Is the set closed under the operation?

(c) Is the operation commutative?

(d) Do you think the operation is associative? Use the operation table to check several examples.

(e) Is there an identity element for the operation ANTH ?

(f) Does each element of the set have an inverse under the operation ANTH ?

8. Let sets A, B, and C be defined as follows:

$A = \{1,2,3,4,5\}$; $B = \{3,4,5,7,8\}$; $C = \{1,3,5,8,9\}$.

(a) Show that the operation $\cup$ (union) distributes over the operation $\cap$ (intersection). That is, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Answers to Exercises

1. One less than the sum of the two numbers. $a \circ b = (a + b) - 1$.
   Note: This can also be written as $(a - 1) + b$ or $a + (b - 1)$.

2. The average of the two numbers. $a \perp b = \frac{a + b}{2}$.

3. The larger of the two numbers if the numbers are different; that number if they are the same.

4. The sum of twice the first number and the second number. $a \sim b = 2a + b$.

5. Twelve minus the sum of the two numbers. $a \sqcup b = 12 - (a + b)$.

6. a 7. 3 8. b 9. 3 10. Yes 11. Yes

12. Yes. No; $7 \odot 3 = 5$ whereas $3 \odot 7 = 3$. You need find only one counterexample to show that commutativity does not hold.

13. No; $c \triangle b = c$ whereas $b \triangle c = a$.

14. Closed, commutative, and associative. (It is understood here that "choose the smaller number" means to select that number if both are the same. That is, 32 combined with 32 produces 32.)

15. Closed and associative; not commutative.

16. b 17. e 18. c 19. Yes 20. Yes 21. Yes: C

22.

Element	a	b	c	d	e
Inverse	e	d	c	b	a

23. 7 24. 10 25. Yes; 12

26. The inverse of 5 is 7 since $5 + 7 = 12$, the identity element for addition.

27. No. For example, $B * (C \circ D) = B * C = D$, whereas $(B * C) \circ (B * D) = D \circ A = A$.

28. Yes. For example, $B \circ (C * D) = B \circ A = A$, and $(B \circ C) * (B \circ D) = C * D = A$.

29. The identity element for $*$ is A.
    The identity element for $\circ$ is B.

Chapter 5

INTRODUCING NEW NUMBERS

Introduction

In this chapter we shall examine in detail some of the different number systems that are encountered in the seventh grade. In some respects the treatment will be that as given in a seventh grade course and in other respects the treatment will be a bit more advanced. Though seventh grade mathematics does not normally include a study of negative numbers we shall introduce them in this chapter. There are three reasons for doing so. (a) The introduction of negative numbers is in many respects similar to the introduction of rational numbers and thus strengthens our understanding of this process. (b) Negative numbers are commonly introduced in the eighth grade and junior high school teachers either teach eighth grade or wish to be knowledgeable in the subject matter their students will learn in the following years. (c) Some youngsters will have met the negative numbers in earlier grades, and we may expect to have more such youngsters in the future.

The development of the real number system that we are about to trace in this and the next four chapters is a remarkable achievement of the human mind. These chapters will present the result of over four thousand years of human thought. In our modern age there are many ways in which this development may be carried out. We shall begin with the counting numbers.

5.1 The Counting Numbers and the Whole Numbers

Although the counting numbers are exceedingly abstract, they do not frighten us, for we are very familiar with them. At this particular time let us accept the counting numbers and some of their properties and build on them. The properties of the counting numbers we wish to build on in the beginning are properties of binary operations. The binary operations, addition and multiplication, of the counting numbers were introduced by man to enable him to make greater use of the counting numbers. These binary operations turn out to have some very useful properties. Both addition and multiplication are binary operations which are closed, commutative, and associative. That there are two binary operations which have these three properties is itself useful and interesting, but the utility and interest is much increased by the fact that these operations are inter-related. For the counting numbers we have the distributive property: $a(b + c) = ab + ac$.

What do we mean when we say a property holds? For example, let us look at multiplication. What do we mean when we say multiplication is commutative? Certainly no one has verified all the possible products; it is most unlikely that the product

$$987685948329573 \times 897869697857463957362$$

has ever been computed and equally as unlikely that the product

$$897869697857463957362 \times 987685948329573$$

has been computed. Nevertheless, we assert with complete confidence that the products are the same. We fearlessly make this assertion because we may derive it from our definition of multiplication. In the abstract systems of Chapter 4 we decided that an operation is commutative by examining a table. The table serves as the definition of the operation; it tells you how to operate on two of the elements of the system to produce a resulting element. From the table, which is the definition of the operation, we derive the properties of the operation. Thus, in Chapter 4 a system was shown to be commutative by examining a table.

For the counting numbers there are too many elements to exhibit a multiplication table. To investigate the properties of multiplication we must go back to a definition. Multiplication of counting numbers is best defined in terms of sets, though sometimes it is done as repeated addition. To show that multiplication of counting numbers is commutative we go back to the definition and derive, logically, that the property holds. Such a development of the counting numbers may be found in many sources.

The counting number 1 has a property shared by no other counting number. With respect to multiplication we have

$$1 \cdot a = a \cdot 1 = a$$

where a represents any counting number. In Chapter 4 we learned to call an element with this property an identity element. Since 1 is an identity element with respect to multiplication, it is called a multiplicative identity.

Class Exercises

1. We know that 2 is another name for 1 + 1. Use this fact and a property of the operations on the counting numbers to show that

$$2 \cdot 2 = 2 + 2 .$$

2. The distributive law tells us $3(5 + 6) = 3 \cdot 5 + 3 \cdot 6$. Why is it true that $3(5 + 6 + 8) = 3 \cdot 5 + 3 \cdot 6 + 3 \cdot 8$?

3. What properties of the counting numbers are used to show that $5 \cdot (6 \cdot 9) = 9 \cdot (6 \cdot 5)$?

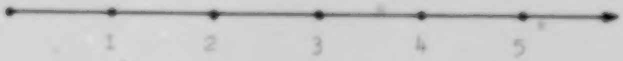
4. Using the properties of the counting numbers, show that $(3 + 4) + (5 + 6) = \big((6 + 4) + 5\big) + 3$.

5. Among the counting numbers an additive identity would be a counting number x with the property that

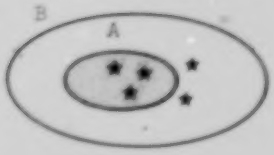
$$x + a = a + x = a$$

for any counting number a. Do the counting numbers have an additive identity?

The properties we have referred to above are properties of two very special binary operations, addition and multiplication. The counting numbers have other properties that are equally interesting and at least as basic. Among these is the property of order. Given two counting numbers, we say that they are equal or that one is greater than another (or smaller). We may use the number line to represent the order that exists among the counting numbers.



Of course, the idea of order for counting numbers exists independently of their number line representation. For example, if set A is a proper subset of the finite set B ($A \subset B$ where $A \neq B$), then the number of elements in set A is less than the number of elements in set B.



Small children learn this relationship long before they learn to use either numerals or the number line. A child knows that five pieces of candy are better than two long before he knows the meaning of the symbols in the sentence "$5 = 2 + 3$."

When we adjoin to the set of counting numbers the number 0, we call the collection thus obtained the set of whole numbers. The whole numbers share with the counting numbers many arithmetical properties. For the whole numbers there are two binary operations, addition and multiplication, that are closed, commutative and associative. The distributive law holds, connecting the two operations. The adjunction of the number 0 provides an additive identity to the set. Thus, with respect to addition we have

$$0 + a = a + 0 = a$$

where $\underline{a}$ represents any counting number.

5.2 Positive Rational Numbers

While the counting numbers have many desirable features, they also suffer from numerous serious deficiencies. There are many elementary questions that we would like to answer, questions that may be asked with counting numbers, but that cannot be answered with counting numbers. Two boys wish to share equally five pieces of candy; how many pieces should each boy receive? Gasoline is 30 cents a gallon; how much gasoline may be purchased for $2.00 ? A man is able to walk ten miles in four hours; how far can he walk in one hour? These questions and many others are reasonable ones that we wish to answer. However, we have learned to go outside the system of counting numbers to find the answers. Most of us have learned to do this in a piecemeal fashion. We learn about two equal parts of a cake, two equal parts of an apple, and so on, eventually coming to the concept of the number we name $\frac{1}{2}$. The same process leads us to concepts for $\frac{1}{4}$ and $\frac{1}{3}$ and other unit fractions.

In this chapter we wish to give a systematic introduction to rational numbers. To this end we abstract from the problems which lead to rational numbers their common feature. Though rational numbers seem to have evolved in many diverse ways, there is a common feature. Indeed all positive rational numbers are solutions of equations stated in terms of counting numbers. The equations

$$2x = 5$$
$$30x = 200$$
$$4x = 10$$

are stated with counting numbers. However, the solutions to the equations given above are not counting numbers. The solutions of these equations provide the answers to the questions of the preceding paragraph. Thus, while the questions seem diverse, the equation approach points out their similarity.

If we restrict ourselves to counting numbers, we would have to say that the solution set of each of the equations $2x = 5$, $30x = 200$, and $4x = 10$ is the empty set. Of course, we may write equations that have nonempty solution sets in terms of counting numbers.

$$3x = 6$$

$$8x = 24$$

$$11x = 341$$

Our state of mind at this point may be compared to a carpenter who has a rule marked in inches without further subdivisions. He is able to work as long as the measurements he needs are full inches. Since most lumber does not measure a whole number of inches he is soon apt to run into trouble. Our mythical carpenter may well do what many small children and some adults do; invent markings. To transfer measurements, a child will ignore the markings between the inch markings on his ruler and make a pencil mark. Real carpenters with real rules will also invent markings. Listen carefully to good carpenters talking and you will hear such things as "a fat eighth and three sixteenths." Their rules are not adequate for their needs, and they invent new quantities, perhaps not very precise, but sufficient for their needs.

There are many ways and levels through which rational numbers may be introduced: through experience, as children learn them, or through ordered pairs, as some texts and most mathematicians do them. We will take a middle course that ties in with seventh grade SMSG texts.

We will "invent" numbers to serve us better in certain real life situations. The concepts of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{12}$, and so on, are introduced physically using candy bars, apples, pies, glasses of milk, and the like. A candy bar split into two equal pieces conveys the idea of $\frac{1}{2}$ to a child. To us the equation $2x = 1$ will carry the same concept of $\frac{1}{2}$.

A pie is cut into four equal pieces, one of which is then eaten. How much remains? Again to us, the concept is probably clear through the equation $4x = 3$. However, the child needs the physical examples to strengthen the concept of $\frac{3}{4}$. As the child develops, we point out to him that $\frac{3}{4}$ is the name of a number with the property that $4 \cdot \frac{3}{4} = 3$. In other words, it is the solution to the equation $4x = 3$.

Likewise, $\frac{5}{12}$ is a name for the number with the property that $12 \cdot \frac{5}{12} = 5$. It is the solution to the equation $12x = 5$.

Actually, we invent mentally a class of numbers that are solutions of equations in the form

$$bx = a \quad (a, b \text{ counting numbers}).$$

Once this is assimilated, fractions, rational numbers, and their relationship to each other lose their mystery.

We have mentally invented numbers that are solutions of equations of this special form. To continue our discussion it will certainly be convenient to have names for these numbers. Collectively we call them the positive rational numbers. The individual numbers are solutions of equations and can be named from the equations. The number that is a solution of $2x = 1$ is named $\frac{1}{2}$. The number that is the solution of $3\frac{1}{2}x = 35$ is named $\frac{35}{3\frac{1}{2}}$, and in general, the number that is a solution of $bx = a$, a and b counting numbers, is named $\frac{a}{b}$. In each case we introduce a symbolic name, a fraction, for the concept of a rational number.

Class Exercises

6. For each equation, give the solution in the form $\frac{a}{b}$, with a and b whole numbers.

(a) $3x = 11$ (c) $10x = 35$

(b) $6x = 15$ (d) $8x = 5$

7. Write an equation for which each of the following is the solution.

(a) $\frac{2}{7}$ (c) $\frac{5}{12}$

(b) $\frac{1}{4}$ (d) $\frac{30}{100}$

Let us pause to reflect on this introduction to positive rational numbers. The child learns about rational numbers through physical experiences. Most texts build upon these experiences to show that a rational number is the solution of an equation. Thus, a child will agree that $\frac{3}{4}$ has the property that $4 \cdot \frac{3}{4} = 3$. We have adopted a different view. We started with the equation and introduced the solution. The end result of the two methods will be the same.

We have invented some numbers and given them names. So far they have only one attribute; they may be used with counting numbers to make true sentences. We wish to take the positive rational numbers and use them to form an algebraic system. This will be done in Chapter 6 where two binary operations will be introduced on the set of positive rational numbers. Though we wish to give an abstract development of rational numbers, we shall not ignore our previous, less formal, knowledge. Rather we shall use this knowledge to suggest the direction of our abstract approach.

5.3 Equivalent Fractions

Some of the problems that concern us here should definitely not be made the concern of junior high school students. Statements that those students accept without hesitation will be examined in detail here. The development is given for the teacher, to help shed light on the structure of the rational number system.

From previous experience with rational numbers, everyone accepts the statement that $\frac{1}{2}$ and $\frac{2}{4}$ name the same rational number. Let us see how this conclusion may be reached without cutting a cake into four parts. From our notation, $\frac{1}{2}$ names a solution of the equation

$$2x = 1$$

while $\frac{2}{4}$ names a solution of the equation

$$4y = 2.$$

To avoid prejudging the matter, we have used x in one equation and y in the other, thus in no way implying the two solutions are necessarily the same. The equation $2x = 1$ has a solution which we have named $\frac{1}{2}$; that is, $2 \cdot \frac{1}{2}$ and 1 are two names for the same number and we write

$$2 \cdot \frac{1}{2} = 1.$$

Since $2 \cdot \frac{1}{2}$ and 1 name the same number, the products $2 \cdot (2 \cdot \frac{1}{2})$ and $2 \cdot 1$ will also name the same number:

$$2 \cdot (2 \cdot \frac{1}{2}) = 2 \cdot 1 = 2.$$

Students will readily accept that this implies:

$$(2 \cdot 2) \cdot \frac{1}{2} = 2$$

$$4 \cdot \frac{1}{2} = 2.$$

But the last equation above is in the form of $4y = 2$.

Thus, it is concluded that the solution of

$$2x = 1$$

is also the solution of

$$4y = 2 ;$$

or that $\frac{1}{2}$ and $\frac{2}{4}$ name the same rational number.

If we carefully look at this reasoning, we see a serious gap. We have fallen into the error of stating

$$(2 \cdot 2) \cdot \frac{1}{2} \overset{?}{=} 2 \cdot (2 \cdot \frac{1}{2}).$$

We have inadvertently used the associative law in a situation where there is no justification for its validity.

Let us pause once again for some reflection. We are attempting to adopt a state of mind in which we invent for ourselves the positive rational numbers and some properties of these new numbers. To do this we must be careful that we do not use a property that is not of our making. We do not regard ourselves as completely free in our invention for experience has taught us that certain properties of binary operations are most useful. Thus, we shall aim for a system with two binary operations that are associative and commutative. Furthermore, we shall want these operations, if possible, to be connected by a distributive law.

Thus, we want to construct a system in which $2 \cdot (2 \cdot \frac{1}{2}) = (2 \cdot 2) \cdot \frac{1}{2}$. There is no guarantee that such can be done. However, from our previous arguement we do know it will be impossible to have an associative binary operation unless we agree that $\frac{1}{2}$ and $\frac{2}{4}$ name the same number.

While the teacher should be aware of the developments of the preceding paragraphs, their content is not appropriate for junior high school students.

By the same reasoning we are led to the conclusion that $\frac{2}{3}$, $\frac{8}{12}$, $\frac{18}{27}$, and $\frac{32}{48}$ should name the same number. That is, the solutions of each of the following equations are equal:

$$3x = 2 \qquad 27z = 18$$
$$12y = 8 \qquad 48x = 32.$$

Before stating a formal definition of equality let us look at another example. The numbers $\frac{4}{36}$ and $\frac{3}{27}$ are solutions of the equations $36x = 4$ and $27y = 3$, respectively; $36 \cdot \frac{4}{36} = 4$ and $27 \cdot \frac{3}{27} = 3$. To compare these two statements, let us multiply the first equation by 27 and the second equation by 36:

$$27 \cdot (36 \cdot \frac{4}{36}) = 27 \cdot 4$$

$$36 \cdot (27 \cdot \frac{3}{27}) = 36 \cdot 3.$$

The numbers 27 and 36 were chosen to give uniformity to the left-hand sides of the two statements. Both now contain the product of the two factors 27 and 36 times the respective numbers $\frac{4}{36}$ and $\frac{3}{27}$. The right-hand sides, $27 \cdot 4$ and $36 \cdot 6$, both name 108. Thus, we see that $\frac{4}{36}$ and $\frac{3}{27}$ must both be solutions of the same equation, $(27 \cdot 36) \cdot x = 108$, and hence should be called names for the same number. How may we decide when two symbols name the same rational number?

Definition: The symbols $\frac{a}{b}$ and $\frac{c}{d}$ name the same rational number if it is true that:

$$bdx = ad$$

and

$$bdx = bc$$

are the same equation.

This definition is applied in several examples that follow:

Example: Do the symbols $\frac{18}{64}$ and $\frac{27}{96}$ name the same rational number? By direct application of the definition we get

$$(64 \cdot 96)x = 18 \cdot 96 \text{ or } 6144x = 1728$$

and

$$(64 \cdot 96)x = 64 \cdot 27 \text{ or } 6144x = 1728.$$

Since the equations are identical, we conclude the two named rational numbers are equal.

Example: Do $\frac{112}{64}$ and $\frac{602}{300}$ name the same rational number?
We examine the equations

$$(64 \cdot 300)x = 112 \cdot 300 \text{ or } 19{,}200x = 33{,}600$$

and

$$(64 \cdot 300)x = 64 \cdot 602 \text{ or } 19{,}200x = 38{,}528.$$

As these equations are not identical, we conclude the two named rational numbers are not equal.

Notice that in each case the actual test of equality is made by comparing equations. The teacher should understand that this development is given here to emphasize the importance of the equation approach to defining rational numbers. On the other hand, it is apparent from the above example that the test of equality for the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ can be made simply by comparing the products ad and bc for equality since these are the products that appear on the right side of the equations. Hence, it is more common for the junior high school student to compare rational numbers using the following definition:

> The symbols $\frac{a}{b}$ and $\frac{c}{d}$ name the same rational number if and only if $ad = bc$.

The teacher should see clearly the comparison between this definition and the previous one as it will help to give background and understanding to the teaching of rational numbers and proportions in the junior high school.

Symbols such as $\frac{2}{3}$, $\frac{9}{12}$, and $\frac{11}{17}$ name rational numbers since each is the solution to an equation in the form $bx = a$ where a and b are counting numbers. Symbols in the form $\frac{a}{b}$ that represent the indicated quotient of two quantities are called fractions. When convenient we use the well-known terminology "numerator, denominator" with fractions. If in the fraction $\frac{a}{b}$ both a and b are counting numbers, then the fraction names a positive rational number.

We name rational numbers with fractions; each rational number has many fractional names. Fractions that name the same rational number are called equivalent fractions. Now that the language of fractions and rational numbers has been developed we will not hesitate to say: The number $\frac{1}{2}$ in place of the number which is named by $\frac{1}{2}$.

From Class Exercise 11 we see that every counting number is a positive rational number. (The set of counting numbers is a subset of the set of positive rational numbers.) Counting numbers have many names; some of the names are fractions.

Class Exercises

Note: Exercises 9 through 12 are an essential part of the development of this section.

8. (a) Do $\frac{3}{4}$ and $\frac{9}{12}$ name the same rational number?

   (b) Do $\frac{31}{64}$ and $\frac{85}{175}$ name the same rational number?

9. We know $2 \cdot 100 = 5 \cdot 40$. May we conclude from this statement that $\frac{2}{5}$ and $\frac{40}{100}$ name the same number?

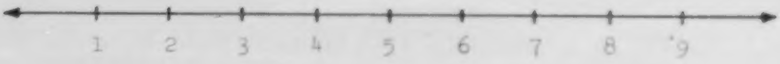
10. If $ad = bc$ (a, b, c, d counting numbers), may we conclude that $\frac{a}{b}$ and $\frac{c}{d}$ name the same number?

11. Do the symbols 4 and $\frac{4}{1}$ name the same number?

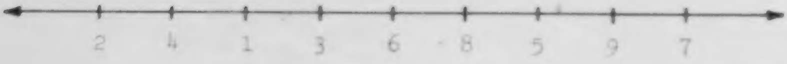
12. Use the definition to show that $\frac{1}{3}$ and $\frac{11}{33}$ name the same rational number. The equation for $\frac{1}{3}$ may be written as $11 \cdot 3x = 11 \cdot 1$. Does this equation show that $\frac{1}{3}$ and $\frac{11}{33}$ represent the same number?

5.4 Order

A valuable representation of the counting numbers is the number line. Can every positive rational number be represented as a point of a number line? The counting numbers are usually represented as follows:



There are other ways to represent the counting numbers on a line as in this figure:



It is clear that every counting number may be represented as a point of the line in this pattern. Nevertheless, we normally reject this representation. It is rejected as it does not contain the information on order indicated by the normal representation. The concept of order also exists for the rational numbers and we would like to indicate this with a number line representation of rational numbers. Order is another aspect of rational numbers of which we have intuitive ideas. Let us make the intuitive ideas, gained from experience, precise.

The numbers $\frac{1}{2}$ and $\frac{3}{4}$ are solutions of the equations $2x = 1$ and $4x = 3$, respectively. We see that $\frac{1}{2}$ and $\frac{3}{4}$ are not equal (meaning they are not names for the same number) since the equations

$$(2 \cdot 4)x = 1 \cdot 4 \qquad \text{or} \qquad 8x = 4 \qquad (x = \tfrac{1}{2})$$

and

$$(2 \cdot 4)x = 2 \cdot 3 \qquad \text{or} \qquad 8x = 6 \qquad (x = \tfrac{3}{4})$$

are not identical.

Since $\frac{1}{2}$ is a solution of $2x = 1$ and of $8x = 4$, while $\frac{3}{4}$ is a solution of $4x = 3$ and of $8x = 6$, we have $8 \cdot \frac{1}{2} = 4$ and $8 \cdot \frac{3}{4} = 6$. Reasoning informally, we might say eight times three-fourths is more than eight times one-half, and hence that three-fourths is greater than one-half.

Let us look at another example. The numbers $\frac{13}{23}$ and $\frac{9}{17}$ are solutions of the equations $23x = 13$ and $17x = 9$, respectively. Do the fractions $\frac{13}{23}$ and $\frac{9}{17}$ name the same number? We examine the equations

$$(23 \cdot 17)x = 13 \cdot 17 \qquad \text{or} \qquad 391x = 221$$

and

$$(23 \cdot 17)x = 23 \cdot 9 \qquad \text{or} \qquad 391x = 207.$$

The equations are not equal since 221 does not equal 207. Therefore, we conclude that $\frac{13}{23} \neq \frac{9}{17}$. Further, since 391 times $\frac{13}{23}$ is greater than 391 times $\frac{9}{17}$, we reason intuitively that $\frac{13}{23}$ is greater than $\frac{9}{17}$.

We see from these last two examples that the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ can be ordered by comparing the products ad and bc.

This reasoning guides us to a formal definition.

Definition: Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers with a, b, c, and d counting numbers. If $ad > bc$, we say $\frac{a}{b} > \frac{c}{d}$, read "$\frac{a}{b}$ is greater than $\frac{c}{d}$."

The reader may complain that the definition is difficult to remember. Some practice with it will help and one may always return to the test for equality and reason as we did above.

Class Exercises

Note: Exercises 15 and 16 are an essential part of the development of this section.

13. Insert in the box the proper sign $<$, $>$, or $=$ to make true statements.

(a) $\frac{3}{4} \square \frac{9}{16}$ (c) $\frac{101}{12} \square \frac{67}{8}$

(b) $\frac{27}{81} \square \frac{18}{55}$ (d) $\frac{10001}{59} \square \frac{63}{2}$

14. Insert in the box the proper sign $<$, $>$, or $=$ to make true statements.

(a) $\frac{18}{25} \square \frac{9}{11}$ (d) $\frac{18}{25} \square \frac{108}{132}$

(b) $\frac{18}{25} \square \frac{27}{33}$ (e) $\frac{9}{11} \square \frac{27}{33}$

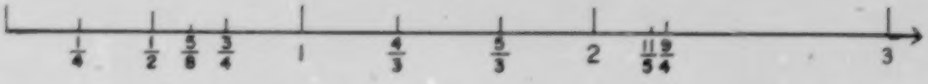
(c) $\frac{18}{25} \square \frac{90}{110}$ (f) $\frac{9}{11} \square \frac{90}{110}$

(g) $\frac{9}{11} \square \frac{108}{132}$

15. Let a, b, and k be counting numbers. Show that $\frac{a}{b} = \frac{ak}{bk}$.

16. Let $\frac{e}{f}$ and $\frac{c}{d}$ be fractions such that $\frac{c}{d} < \frac{e}{f}$. Let k be a counting number and show that $\frac{ck}{dk} < \frac{e}{f}$.

Exercises 15 and 16 partially show that the order relation of the rational numbers does not depend on the particular fractional representation. Now that a definition of order has been introduced we may systematically make rational numbers correspond to points on a number line. As an example let us search for a point corresponding to $\frac{9}{4}$. It is readily shown that $2 = \frac{2}{1} < \frac{9}{4}$ while $\frac{9}{4} < \frac{3}{1} = 3$. That is, we would like the point representing $\frac{9}{4}$ to be between the points whose coordinates are 2 and 3. Which point between 2 and 3 should we choose? Here we fall back upon our idea of measure on a line. (See Chapter 12.)

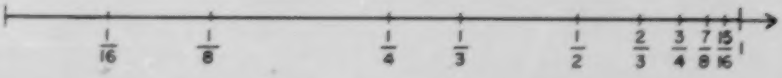


Here the visual representation of the positive rational numbers on the number line indicates the relative magnitude of the rational numbers. Each $\frac{1}{4}$ unit corresponds to the same distance.

A more customary approach to the ordering of the rational numbers and to the number line begins with the number line itself. For example, the line between 0 and 1 is divided into four parts of equal length. The end points of the parts are labelled $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. Also, the same segment is divided into five parts of equal length with end points labelled $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1. By inspection, we determine that $\frac{4}{5}$ is greater than $\frac{3}{4}$. This procedure becomes unmanageable for such fractions as $\frac{131}{725}$ and $\frac{654}{4312}$. Thus, we are forced to use the more sophisticated approach of our definition. For children, the geometric or physical introduction is recommended, quickly followed by the algebraic approach.

Class Exercises

17. Here is a possible correspondence of rational numbers and points on a line. Criticize this correspondence.



5.5 Whole Numbers and Rational Numbers

We have confined ourselves to equations of the form bx = a, (a, b counting numbers). Let us extend our horizons and examine equations of the form bx = a where we now let a and b be whole numbers. Though we now have changed the setting of our discussion to include zero, much remains familiar. All the equations involving counting numbers are still with us. There are, however, some new equations. Some examples of these are:

$$\begin{array}{ll} 2x = 0 & 0w = 0 \\ 0y = 17 & 0x = 5 \\ 3z = 0 & 5x = 0 \end{array}$$

Let us look at some of these examples, recalling that the product of 0 and any number is 0. This latter fact makes us rule out equations of the form $0y = 17$; $0x = 5$; $0x = a$ where $a \neq 0$. The reader should become indignant at the suggestion that we rule out the equation $0x = 5$ and should demand that more new numbers be invented so as to solve equations of this form.

Indeed, it would be possible to make up such new elements; however, what are the consequences? We would, of course, have to give up the result that the product of 0 and any number is 0. We would have to give up the distributive law which could no longer hold. The whole structure of arithmetic would collapse. The gain is not worth what would be lost and so we do not allow such invention. Since we now exclude the possibility of solutions to $0x = a$ where $a \neq 0$, we are in essence saying that the corresponding symbol $\frac{a}{0}$ has no meaning.

Equations of the form $0x = 0$ do have solutions. Since the product of 0 and any number is 0 we may substitute any number for x to obtain a true statement. But the equation $0x = 0$ does not define any unique number and so these equations are also ruled out. Thus, the corresponding symbol $\frac{0}{0}$ also has no meaning. Indeed, these last two results lead us to the statement: We cannot divide by zero.

Finally, there are the equations of the form $2x = 0$, $3z = 0$, $5x = 0$, $bx = 0$, (b a counting number). These equations have 0 as a solution. With our original notation for fractions, we denote the solution of $2x = 0$ with $\frac{0}{2}$ and the solution of $ax = 0$, $a \neq 0$, with $\frac{0}{a}$.

Class Exercises

18. Use the definition to show that the fractions $\frac{0}{2}$ and $\frac{0}{5}$ name the same rational number and that this number must be identified with 0.

19. If the product of two numbers is 0 in multiplication mod 5, must one of the factors be zero? Answer the same question for multiplication mod 4.

5.6 The Integers

In Section 2 of this chapter we observed that the counting numbers do not provide a system rich enough to contain solutions to equations such as $3x = 4$. This provided the opportunity to introduce some new numbers, the positive rational numbers. If we return again to the counting numbers, we find another class of questions stated in terms of the counting numbers that cannot be answered with counting numbers (or with positive rational numbers). The questions or equations which were used to introduce the rational numbers were multiplicative in nature; now we look at those which are additive.

Here are some questions:

(a) John is now 12 years old. How old will he be 16 years from now?

(b) Mary had 5 Beatle records. She received 4 more for her birthday. How many Beatle records does she now have?

(c) Mrs. Smith has 2 books of trading stamps. She wishes to obtain a three-piece towel set which requires 97 books of stamps. How many more books does Mrs. Smith need?

(d) The constitution requires that the President of the United States be 35 years old. John is now 27. In how many years will he be eligible to be president?

The answers to (a) and (b) are obtained by using the binary operation of addition on the counting numbers. Problems (c) and (d) may also be phrased as addition problems:

What number when added to 2 yields the sum 97 ?

What number when added to 27 yields the sum 35 ?

Put in terms of open sentences, we want the solution sets of the equations (open sentences)

$$2 + x = 97 \quad \text{and} \quad 27 + x = 35.$$

These are the equations we want solved. However, we have learned a systematic attack on such problems through subtraction and we immediately fall back upon it by solving

$$97 - 2 = x \quad \text{and} \quad 35 - 27 = x.$$

There are other questions that may be asked in the framework of counting numbers.

(e) What is the solution set of $9 + x = 4$?

(f) What is the solution set of $5 + x = 5$?

If we confine ourselves to the counting numbers, we would have to answer that the solution sets of (e) and (f) are the empty set. That is, there is no counting number that may be added to 9 to produce the sum 4, nor is there any counting number that may be added to 5 to produce the sum 5. (Recall that 0 is not a counting number.)

This presents a most unsatisfactory situation; some equations like $4 + x = 9$ have nonempty solution sets while others like $9 + x = 4$ have empty solution sets within the framework of the counting numbers. Yet we frequently want solutions to problems that take the form of question (e). For example:

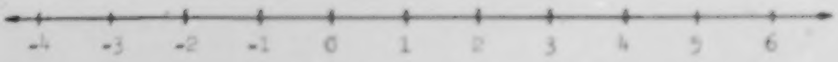
(e') Mary has 9 Beatle records. Her father can tolerate only 4 of them. What can the father do to make the situation tolerable? A drastic solution would have the father destroy 5 records.

(e") Marvin asked for directions. He was told to make a right turn at the fourth light. Through an oversight Marvin went to the ninth light. Is there any way for Marvin to return to the fourth light? (U-turns are allowable.)

Again we wish to develop numbers to answer the questions we can ask. However, there is a difference this time. The student generally has had some experience with the solution of the equation $2x = 1$, while rarely does the student have any experience with a solution of the equation $x + 2 = 1$. Thus, before a study of equations can be successfully started, some informal background experience is useful. This is commonly done by returning to the number line for counting numbers. Recall that certain uniformly spaced points on the line corresponding to the counting numbers are singled out and named 1, 2, 3, ... One other point has been marked and named 0.

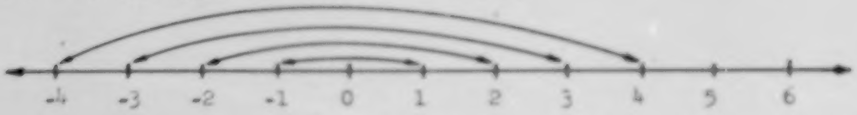
We will now extend the number line to the left. Many devices are used to justify naming points to the left of 0: thermometers, bank accounts, altitude above and below sea level, and distance. Let us simply say that points on one portion of the line have been named and that we wish to name points on the other portion. We could use I, II, III, IV, V, VI, ...

It is more convenient and much more useful to make use of the Hindu-Arabic numerals. In order to be able to differentiate between those naming points to the right of 0 and those to the left of 0 we use the symbol "¯" to denote numerals corresponding to points on the left.

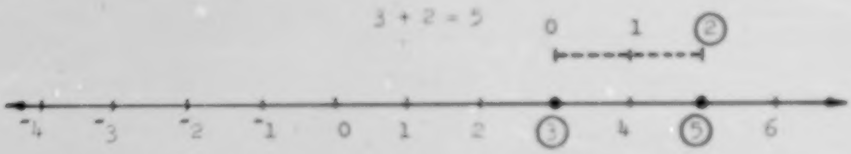


Frequently points to the right of 0 on the number line are named with the symbol "+" to emphasize the distinction between these and the ones on the left of 0. For example, ¯1 names a point one unit to the left of 0 while +1 names a point one unit to the right. These symbols are read "negative one" and "positive one", respectively. Of course, 1 and +1 are just two different ways of naming the number 1 while 1 and ¯1 name two different numbers.

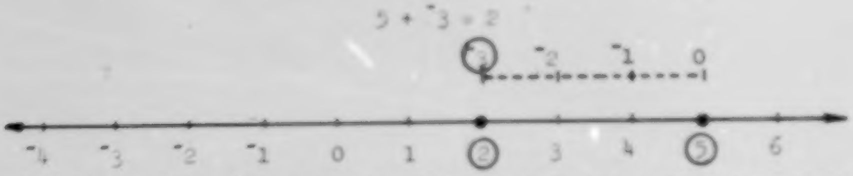
Note that on the number line we have now located a point opposite to each counting number: ¯1 is the opposite of 1, ¯2 is the opposite of 2, and so on. For every counting number b, there is a corresponding negative number, ¯b. The opposite of 0 is 0 itself.



Recall that the number line is admirably suited for describing addition of counting numbers. To add counting numbers we "add" corresponding segments.



Now if we interpret "¯" as meaning we go that amount to the left, we may perform "additions" of other segments (all segments begin at 0).



The figure above indicates the addition using the segment between 0 and 5 and the segment between 0 and ¯3.

Class Exercises

20. Use line segments to perform the following additions:

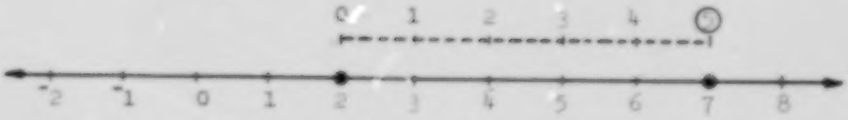
(a) $3 + 4 =$ ______ (d) $^-6 + 5 =$ ______

(b) $4 + {}^-1 =$ ______ (e) $^-2 + {}^-4 =$ ______

(c) $5 + {}^-6 =$ ______ (f) $^-3 + 3 =$ ______

With this interpretation of combining segments we now have a physical method to solve equations of the form $a + x = b$, a and b counting numbers. This is comparable to the aids used to learn about fractions.

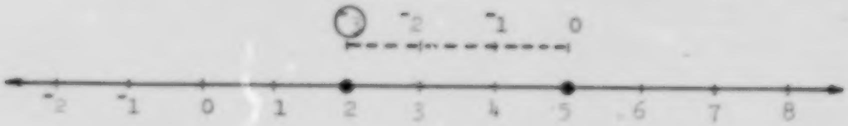
Example: Solve the equation $2 + x = 7$.



$$2 + 5 = 7$$
$$x = 5.$$

In other words, we must move 5 units to the right from 2 in order to reach 7.

Example: Solve the equation $5 + x = 2$.



$$5 + {}^-3 = 2$$
$$x = {}^-3$$

In this case, we must move 3 units to the left from 5 in order to reach 2.

Class Exercises

21. Use the method of the above examples to solve the equations:

(a) $2 + x = 11$ (d) $5 + x = 5$

(b) $5 + x = 6$ (e) $8 + x = 8$

(c) $2 + x = 1$ (f) $8 + x = 2$

We see from the class exercises that solutions to equations of the form $e + x = f$, e and f counting numbers, can be positive, zero, and negative. The collection of all solutions to equations of this form is called the set of integers. Each member of the set is called an integer.

The set of integers is sometimes represented in the following form:

$$I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}.$$

We see that the set of integers consists of:

the set of counting numbers, $\{1, 2, 3, \ldots\}$,
called the positive integers;

the number zero, 0 ;

the opposites of the set of counting numbers,
$\{{}^{-}1, {}^{-}2, {}^{-}3, \ldots\}$, called the negative integers.

The subset of the integers which consists of the counting numbers and the integer 0 is called the whole numbers.

The extended number line naturally introduces an ordering of the integers. Given two integers we locate them on the extended number line and call the one to the right the greater. Thus, we call $2 > {}^{-}4$ and ${}^{-}17 > {}^{-}93$. The latter example sometimes causes uneasiness among students. The essential point is that order on the number line involves direction, which is the extension of the notion of order of the counting numbers.

Insert in the box the proper sign $=$, $>$, or $<$ to make true statements:

(a) ${}^{-}4 \square 4$ (e) $6 + {}^{-}7 \square 7 + {}^{-}6$

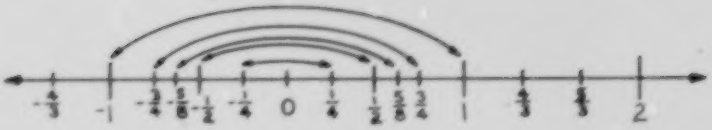
(b) ${}^{-}3 \square {}^{-}2$ (f) $4 + {}^{-}1 \square 3$

(c) ${}^{-}3 \square {}^{-}4$ (g) $5 + {}^{-}7 \square 2$

(d) ${}^{-}11 \square {}^{-}17$ (h) $7 + {}^{-}11 \square 9 + {}^{-}13$

Our discussion of the rational numbers in the previous sections developed only the non-negative rationals. With our knowledge of the integers we now can complete the set of rational numbers by including negative rationals. Recall that the set of whole numbers was extended, by including their opposites, to form the set of integers. In like fashion we will extend the set of positive rational numbers by including their opposites.

Hence, to each positive rational number there is a corresponding negative rational number. Some of these opposites are shown on the number line below.



The complete set of rational numbers now includes all positive rational numbers, all negative rational numbers, and zero.

5.7 Ordered Pairs

It should be clear from our discussion in this chapter that each positive rational number can be introduced simply as an ordered pair of counting numbers; that is, by a pair of counting numbers with the elements of the pair distinguished as to first and last numbers. The notation (a,b), a and b counting numbers, is used to denote an ordered pair. Thus, we can represent a rational number like two-thirds by an ordered pair $(2,3)$ as well as by a fraction $\frac{2}{3}$. Likewise, nine-halves can be represented as $(9,2)$ as well as $\frac{9}{2}$. With this ordered pair notation, it is clear which member of the pair is the first member and which is the last. The ordered pair $(2,3)$ names a different number from the ordered pair $(3,2)$ just as the fractions $\frac{2}{3}$ and $\frac{3}{2}$ name different numbers.

We say that two pairs, (a,b) and (c,d), are equivalent if $ad = bc$. A rational number is a set of all equivalent ordered pairs. The ordered pairs

$$(3,4), \quad (9,12), \quad \text{and} \quad (75,100)$$

all represent the same rational number as do the corresponding fractions

$$\frac{3}{4}, \quad \frac{9}{12}, \quad \text{and} \quad \frac{75}{100}.$$

The method of Section 2 that develops the rationals by the equation method is essentially that of SMSG while the method of ordered pairs of this section is suggested in some other elementary texts. We believe the equation method to be the most satisfactory for young students, but for completeness include the ordered pair method.

In the equation method the experience of the student is used to motivate a belief in the solution of certain equations. An ordered pair approach also uses the student's experience, but in a more formal way.

Class Exercises

23. Write each fraction using the ordered pair notation.

(a) $\frac{3}{8}$ (c) $\frac{2}{3}$

(b) $\frac{9}{16}$ (d) $\frac{1}{100}$

24. Indicate which ordered pairs name the same rational number.

(a) (3,5) and (6,10) (c) (6,6) and (9,9)

(b) (5,7) and (7,5) (d) (5,4) and (15,12)

We have seen how the positive rational numbers can be defined in terms of solutions to equations of the form $bx = a$ while the integers can be defined in terms of solutions to equations of the form $b + x = a$, where in both cases a and b are counting numbers.

To make the analogy between the introduction of positive rational numbers and the integers complete, we may do the following. Let us say that we mentally construct a solution to the equation $7 + x = 3$. We know that among the counting numbers there is no solution. However, we have a physical interpretation of a solution on the number line. Suppose we denote the solution of $7 + x = 3$ by $3 \# 7$ (say "three sharp seven"). In so doing, we are saying that $3 \# 7$ has the property that

$$7 + (3 \# 7) = 3.$$

Just as $\frac{3}{7}$ may be interpreted as representing 3 of 7 equal parts of a circle, we may think of $3 \# 7$ as a name of the point obtained by performing the addition $3 + {}^{-}7$ on the number line.

Are there other equations that have the same solution as $7 + x = 3$? Consider the equation $34 + x = 30$. We can denote its solution by $30 \# 34$ since

$$34 + (30 \# 34) = 30.$$

On the number line the solution may be obtained by performing the addition $30 + {}^-34$. However, we find on the number line that the point named from the addition $30 + {}^-34$ is the same as that named from the addition $3 + {}^-7$. Thus, our invented names for these solutions, $3 \# 7$ and $30 \# 34$, must represent the same number, ${}^-4$. In other words,

$$3 + {}^-7 = 30 + {}^-34 = {}^-4.$$

The solutions to $7 + x = 3$ and $34 + x = 30$ are equal. Here we have the same situation as for the positive rational numbers. There are many equations that lead to numbers we would like to call the same. This is handled in precisely the same way. We agree that different symbols may be different names for the same number. For fractions that name solutions of multiplicative equations the determination is made in terms of products of counting numbers. Two fractions $\frac{a}{b}$, $\frac{c}{d}$, name the same number if and only if $ad = bc$. For additive equations, equations of the form $a + x = b$, a and b counting numbers, a determination is made in terms of sums of counting numbers. If $(a + {}^-b)$ and $(c + {}^-d)$ are names of solutions of $b + x = a$ and $d + y = c$, respectively, (a, b, c, d counting numbers), then they represent the same number if and only if

$$a + d = c + b.$$

While the notation $3 \# 7$, $30 \# 34$, and $7 \# 3$ is, of course, not standard mathematical notation, it does help to emphasize to the reader that the integers can also be treated as ordered pairs of whole numbers. The ordered pairs used in $3 \# 7$ and $30 \# 34$ represent the same integer while the ordered pairs in $3 \# 7$ and $7 \# 3$ do not.

Class Exercises

25. Indicate which of the two sums represent the same integer.
    (a) $3 + {}^-4$ and $54 + {}^-51$
    (b) $17 + {}^-28$ and $84 + {}^-96$
    (c) $12 + {}^-12$ and $17 + {}^-17$
26. Indicate if the two symbols name the same integer.
    (a) $(11 + {}^-7)$ and 4 (b) $(81 + {}^-99)$ and ${}^-18$ (c) $(10 + {}^-12)$ and ${}^-2$

5.8 Historical Note

In introducing the positive rational numbers before the integers we have followed historical precedence. Sometime before 1700 B.C. the Egyptians were using positive rational numbers. We have been able to date this knowledge due to the discovery of several Egyptian manuscripts. The best known of these is called the Rhind papyrus. An excellent outside assignment would be a report on the Rhind papyrus. (The Encyclopedia Britannica, 11th edition, is a fine source.) Though less well known, the Babylonians of 4,000 years ago also had a knowledge of rational numbers.

The development of the integers came much later, as far as we know. When discussing the origin of ideas one must remember that civilization is not static. Many great nations with complex societies have come and gone. Relics of these people are hard to find, if indeed any relics still exist. Knowledge and libraries are always the targets of despots. The library at Alexandria was wantonly destroyed. It is said that Shih Huang Ti, the emperor of China in 221 B.C. ordered all books of learning destroyed. You will be able to supply some modern instances of attempts to destroy knowledge. Nature also conspires against the preservation of knowledge. Manuscripts written on bark do not long survive.

There is evidence that an appreciation of the integers was developing in the fifth century A.D. Another 1000 years were to pass before the integers were completely absorbed. A complete and rigorous development of our number system was not given until the 18th century.

A modern development of the number system would not follow the historical pattern of development. Rather, one would, after introducing the counting numbers, proceed to the integers. From the integers one would develop the rational numbers (positive and negative) and then go on to the system known technically as the real numbers.

Chapter Exercises

1. Show that $4(5 + 6 + 7 + 8) = (4 \cdot 5) + (4 \cdot 6) + (4 \cdot 7) + (4 \cdot 8)$.

2. Show that $(((3 + 4) + 9) + 5) + 2 = 3 + (4 + (9 + (5 + 2)))$.

3. Give two other symbols which name the same rational number as does $\frac{51}{64}$.

4. What is improper about an improper fraction?

5. Does $\frac{11}{5}$ name a solution of $35x = 77$? Does $\frac{7}{9}$ name a solution of $21x = 27$?

6. Show that $\frac{3}{4}$ and $\frac{15}{20}$ name the same rational number. Show that $\frac{3}{4}$ and $\frac{27}{36}$ name the same rational number.

7. What counting number may be used to name $\frac{6}{1}$? $\frac{2}{1}$? $\frac{103}{1}$?

8. Is there a counting number that may be used to name $\frac{6}{2}$? $\frac{8}{4}$? $\frac{93}{3}$? $\frac{ak}{k}$?

9. Order the following rational numbers beginning with the smallest:

   $\frac{3}{2}$, $\frac{3}{17}$, $\frac{3}{1}$, $\frac{3}{6}$, $\frac{3}{8}$, $\frac{3}{5}$.

10. Order the following rational numbers beginning with the smallest:

    $\frac{2}{7}$, $\frac{9}{7}$, $\frac{6}{7}$, $\frac{7}{7}$, $\frac{18}{7}$, $\frac{5}{7}$, $\frac{14}{7}$.

11. Order the following rational numbers beginning with the smallest:

    $\frac{8}{9}$, $\frac{11}{12}$, $\frac{19}{20}$, $\frac{14}{15}$, $\frac{99}{100}$.

12. Which of the following statements are true?

    (a) The integers are opposites of the counting numbers.

    (b) Zero is an integer.

    (c) The set of whole numbers includes only the positive integers.

    (d) The integer $^{-}17$ is less than the integer $^{-}15$.

    (e) Every integer can be expressed as the solution of an equation in the form $a + x = b$, a and b counting numbers.

Answers to Class Exercises

1. Write $2 \cdot 2$ as $2(1 + 1)$ and use the distributive property to arrive at $2 \cdot 2 = 2 + 2$.

2. The distributive law, $a(b + c) = ab + ac$, on the left side tells us something about the sum of two counting numbers. The product $3(5 + 6 + 8)$ involves the sum of three numbers. It is still possible to use the distributive property, for $5 + 6 + 8$ means $(5 + 6) + 8$. That is, $5 + 6 + 8$ may be regarded as the sum of two numbers, one named $5 + 6$ and the other named 8. Now write $3(5 + 6 + 8)$ as $3\big((5 + 6) + 8\big)$. From the distributive property this may be written as $3(5 + 6) + 3 \cdot 8$. One other application of the distributive property gives the required result.

3. There are many ways to do this problem, all requiring the use of the properties of the counting numbers. Here is one.

$$\begin{aligned} 5 \cdot (6 \cdot 9) &= (5 \cdot 6) \cdot 9 && \text{Associative property of multiplication} \\ &= 9 \cdot (5 \cdot 6) && \text{Commutative property of multiplication} \\ &= 9 \cdot (6 \cdot 5) && \text{Commutative property of multiplication} \end{aligned}$$

4. This problem may also be done in many orders. To group 4, 5, and 6 together we think of $5 + 6$ as one number and use the associative property $(a + b) + c = a + (b + c)$. All properties used apply to addition.

$$\begin{aligned} (3 + 4) + (5 + 6) &= 3 + \big(4 + (5 + 6)\big) && \text{Associative Property} \\ &= \big(4 + (5 + 6)\big) + 3 && \text{Commutative Property} \\ &= \big(4 + (6 + 5)\big) + 3 && \text{Commutative Property} \\ &= \big((4 + 6) + 5\big) + 3 && \text{Associative Property} \\ &= \big((6 + 4) + 5\big) + 3 && \text{Commutative Property} \end{aligned}$$

5. No. For the counting numbers to have an identity with respect to addition there would have to be a counting number which added to 1 would give the sum 1. (Remember, 0 is not a counting number.) To prove something is true we must show it true in all cases. To show a general statement is not true, it is enough to show that it does not hold in one special case, as we have done here.

6. (a) $\frac{11}{3}$ (b) $\frac{15}{6}$ (c) $\frac{93}{10}$ (d) $\frac{9}{8}$

7. (a) $7x = 2$ (b) $4x = 3$ (c) $12x = 5$ (d) $100x = 50$

8. (a) yes (b) no

9. Yes. The number named by $\frac{2}{5}$ is a solution of $5x = 2$ and $\frac{40}{100}$ is a name for the solution of the equation $100x = 40$. To apply the test we compare the equations

$$100 \cdot 5x = 100 \cdot 2$$

and

$$5 \cdot 100x = 5 \cdot 40 .$$

The multipliers 100, for the first equation, and 5, for the second equation were chosen so that the left-hand sides of both test equations are equal. Thus, to decide if the two equations are the same, we need only compare the two right sides.

The answer also follows directly from the statement: $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$, b and d unequal to zero. (See answer to Exercise 10 below.)

10. Yes. The number named by $\frac{a}{b}$ is a solution of the equation $bx = a$ and $\frac{c}{d}$ is the name for the solution of the equation $dx = c$. To apply the test we compare the equations

$$dbx = da$$

and

$$bdx = bc.$$

The multipliers, d for the first equation and b for the second equation, were chosen so that the two left sides of both test equations are equal. Thus, to decide if the two equations are the same, we need only compare the two right sides. If $ad = bc$, $\frac{a}{b}$ and $\frac{c}{d}$ name the same number.

11. This question cannot be baldly answered yes or no. The symbol $\frac{4}{1}$ is a name for a solution of the equation $1 \cdot x = 4$. The equation $1 \cdot x = 4$ also has a counting number as a solution; namely, $x = 4$. We agree that these two symbols should name the same number. Our intuitive notion of fraction corroborates this agreement; $\frac{4}{1}$ths of a pie would be 4 pies.

12. Yes. The equations for $\frac{1}{3}$ and $\frac{11}{33}$ are, respectively, $3x = 1$ and $33x = 11$. To perform the test of the definition we examine the equations

$$3 \cdot 33x = 1 \cdot 33$$
$$3 \cdot 33x = 3 \cdot 11.$$

Since $33 = 3 \cdot 11$ we see that the equation $3 \cdot 33x = 1 \cdot 33$ may be obtained from $11 \cdot 3x = 11 \cdot 1$ (multiply by 3). The test of the definition requires that we multiply the equation for $\frac{11}{33}$, $33x = 11$, also by 3. Thus, by noting these facts we can assure ourselves that the test is satisfied and save some multiplications.

13. (a) $>$ (b) $>$ (c) $>$ (d) $>$

14. (a) $<$ (b) $<$ (c) $<$ (d) $<$
(e) $=$ (f) $=$ (g) $=$

15. This may be readily seen by using the test of the definition.

The equation $bkx = ak$ has the solution $x = \frac{ak}{bk}$.

The equation $bx = a$ has the solution $x = \frac{a}{b}$.

Multiplying the first by b gives

$$bbkx = bak.$$

Multiplying the second by bk gives

$$bkbx = bka.$$

16. We know from the given information that $cf < de$. It follows that $cf \cdot k < de \cdot k$ which proves the assertion.

17. To satisfy our physical intuition regarding rational numbers, we would like a segment corresponding to $\frac{1}{2}$ to be twice as long as a segment corresponding to $\frac{1}{4}$.

18. We return to the equations $2x = 0$ and $5x = 0$. Multiplying by 5 and 2, respectively, we obtain

$$5 \cdot 2x = 5 \cdot 0 = 0$$
$$2 \cdot 5x = 2 \cdot 0 = 0.$$

As these two equations are the same, $\frac{0}{2}$ and $\frac{0}{5}$ name the same rational number. Moreover, the equations $2x = 0$ and $5x = 0$ have 0 as solution so we identify 0 with $\frac{0}{2}$ and $\frac{0}{5}$.

19. Yes for mod 5. No for mod 4, since in addition to having at least one factor 0 to give a 0 product we also have

$$2 \times 2 \equiv 0 \pmod{4}.$$

20. (a) 7 (b) 3 (c) $^-1$ (d) $^-1$ (e) $^-6$ (f) 0

21. (a) 9 (b) 1 (c) $^-1$ (d) 0 (e) 0 (f) $^-6$

22. (a) <
(b) <
(c) >
(d) <
(e) <
(f) =
(g) <
(h) =

23. (a) (3,8) (b) (9,16) (c) (5,3) (d) (1,100)

24. (a), (c), (d)

25. (c)

26. (a), (b), (c)

Chapter 6

BINARY OPERATIONS

Introduction

In Chapter 5 the rational numbers were introduced and accommodated on the number line. In this chapter binary operations will be defined on the rational numbers and the properties of these binary operations will be investigated. When we formulate the definition of these binary operations we will want the arithmetic of rational numbers to reflect our past experiences. For example, our experience dictates that $\frac{1}{2} + \frac{1}{2}$ should be 1. In an idealized form $\frac{1}{2}$ a pie plus $\frac{1}{2}$ a pie is a pie. This is, of course, idealized; it is extremely difficult to put two halves of a cherry pie together to have a whole pie. We shall also find ourselves motivated by what we regard as desirable features of a number system.

In the last chapter we looked first at the positive rationals as the set of all solutions to equations in the form $bx = a$ where $\underline{a}$ and $\underline{b}$ are counting numbers. We then introduced zero as a rational number by considering all solutions to the equation $bx = 0$ where $\underline{b}$ is a counting number. Last, we took the opposites of all the positive rationals to form the negative rationals. These three sets, the positive rationals, the negative rationals, and zero together form the *set of rational numbers*. The counting numbers, the whole numbers, and the integers are all contained in the set of rational numbers and hence each is a subset of the set of rational numbers.

In naming these numbers we agreed to identify symbols such as 4 with a counting number and fractions such as $\frac{8}{2}$ with a rational number. Though the symbols 4 and $\frac{8}{2}$ have different geneologies, we agree that they name the same number. The words "horse" and "cheveau" have different origins but they name the same animal. A person who speaks both English and French would use the words interchangeably depending upon the situation. When we define binary operations for the rational numbers we shall want the definitions made in such a way that they agree with the known definitions for the counting numbers.

The point of view of the last chapter will also be used in this chapter. We have been assuming that we are inventing rational numbers. We have a certain amount of intuition to guide us and to suggest the final form of our invention. Our knowledge of rational numbers is that gained from taking them as solutions of equations. To proceed, then, we will make extensive use of this defining knowledge. This point of view is different from that given in most texts. For example, in *Mathematics for Junior High School*, Vol. 1, it is assumed that there are rational numbers, that binary operations are defined on them, and that these binary operations have certain properties. In this text, for teachers, we prefer to show that it is not necessary to make these assumptions since they can be shown to follow directly from the definitions of the operations.

6.1 Addition

Let us begin with an introduction of a binary operation, addition. We start with some simple specific cases to illustrate the method we will use. Suppose we wish to find a rational number to be called the sum of $\frac{1}{2}$ and $\frac{1}{2}$. To mathematically motivate this sum we return to our meaning of $\frac{1}{2}$. We think of $\frac{1}{2}$ as a solution of the equation $2x = 1$; that is, $\frac{1}{2}$ has the property that $2 \cdot \frac{1}{2} = 1$. The symbols 1 and $2 \cdot \frac{1}{2}$ are names of the same counting number. As we wish to define $\frac{1}{2} + \frac{1}{2}$, let us try to involve $\frac{1}{2}$ and $\frac{1}{2}$ in a single statement. One way to do this is to write the true statement

$$2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1 + 1 .$$

This statement is true since $2 \cdot \frac{1}{2}$ is another name for 1. Thus, it is meaningful to write $2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$ since it is another name for the sum of the counting numbers 1 and 1. In our treatment as yet, we do not have a meaning attached to the sum $\frac{1}{2} + \frac{1}{2}$. If it is possible to define operations on the rational numbers such that the distributive law holds, then we would be able to obtain from

$$2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 2$$

the statement

$$2 \cdot (\frac{1}{2} + \frac{1}{2}) = 2.$$

If the distributive law is to hold and $\frac{1}{2} + \frac{1}{2}$ is to have a meaning, we must agree that $\frac{1}{2} + \frac{1}{2}$ is the name of a solution of the equation

$$2y = 2 .$$

But we know that the equation $2y = 2$ has $y = \frac{2}{2}$ as a solution. Hence, we shall agree that $\frac{1}{2} + \frac{1}{2}$ and $\frac{2}{2}$ name the same number. Since $\frac{2}{2}$ is a name for the number one, we define the sum $\frac{1}{2} + \frac{1}{2}$ to be 1.

Let us go through this in another simple case. The rational number $\frac{1}{3}$ is a solution of the equation $3x = 1$, or equivalently, $3 \cdot \frac{1}{3} = 1$. To motivate a meaning for $\frac{1}{3} + \frac{1}{3}$, we proceed as before. The definition of $\frac{1}{3}$

tells us that $3 \cdot \frac{1}{3} = 1$. To relate $\frac{1}{3}$ and $\frac{1}{3}$ we write the true statement

$$3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 1 + 1$$

or

$$3(\frac{1}{3} + \frac{1}{3}) = 2 .$$

This tells us that if a distributive law holds we want to call $\frac{1}{3} + \frac{1}{3}$ a solution of the equation $3z = 2$. That is, the sum $\frac{1}{3} + \frac{1}{3}$ should be called $\frac{2}{3}$ as we know $\frac{2}{3}$ is a name for the solution of $3z = 2$.

The two examples above indicate the procedure that shall be used to define addition of rational numbers. Clearly, they were very specialized examples and examples for which the decisions could easily have been made from physical models. Now let us look at something which is less obvious physically. To define the sum of $\frac{2}{3}$ and $\frac{5}{8}$ we may begin as before. The number $\frac{2}{3}$ is a solution of $3x = 2$ and $\frac{5}{8}$ is a solution of $8y = 5$; that is, $3 \cdot \frac{2}{3} = 2$ and $8 \cdot \frac{5}{8} = 5$. To combine $\frac{2}{3}$ and $\frac{5}{8}$ we may try the above method; combine the two equations:

$$3 \cdot \frac{2}{3} + 8 \cdot \frac{5}{8} = 2 + 5 = 7.$$

This time, however, there is a difference. Even with the use of the distributive law we are unable to group together $\frac{2}{3}$ and $\frac{5}{8}$.

What to do? The first step may be to ask why the procedure failed. To answer this question we must be clear on what the procedure was. To go from the statement $3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2$, to the statement $3(\frac{1}{3} + \frac{1}{3}) = 2$ required the use of distributivity. In general, the distributive property is stated as

$$ab + ac = a(b + c).$$

The expression $3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3}$ seems tailor-made to use a distributive property as we have a common factor (multiplier). This is, of course, the reason for the difficulty with

$$3 \cdot \frac{2}{3} + 8 \cdot \frac{5}{8} ;$$

there is no common factor!

Should this technique be abandoned? This question is important and deserves some serious thought before an answer is given. One line of thought might lead us back to the preceding chapter and the introduction of rational numbers and fractions. On several occasions we wished to compare numbers named by fractions. This was done in such a way that the new equations had equal co-efficients.

After this reflection let us return to our problem, and see if this train of thought has been useful. The fraction $\frac{2}{3}$ is the solution of $3x = 2$, so that $3 \cdot \frac{2}{3} = 2$. Also $\frac{5}{8}$ has the property $8 \cdot \frac{5}{8} = 5$. Let us multiply the first equation by 8 and the second equation by 3. (The multipliers are the denominators of the two fractions involved.) These multiplications yield

$$8 \cdot (3 \cdot \frac{2}{3}) = 16$$

$$3 \cdot (8 \cdot \frac{5}{8}) = 15$$

or assuming the associative property,

$$(8 \cdot 3) \cdot \frac{2}{3} = 16$$

$$(3 \cdot 8) \cdot \frac{5}{8} = 15.$$

Now let us add the counting numbers 16 and 15 and multiply the counting numbers 8 and 3.

$$(8 \cdot 3) \cdot \frac{2}{3} + (3 \cdot 8) \cdot \frac{5}{8} = 16 + 15$$

$$24 \cdot \frac{2}{3} + 24 \cdot \frac{5}{8} = 31.$$

Using the distributive property, we get

$$24(\frac{2}{3} + \frac{5}{8}) = 31.$$

Thus, it is seen that $\frac{2}{3} + \frac{5}{8}$ should be a name for the solution of the equation $24z = 31$ and that we should say

$$\frac{2}{3} + \frac{5}{8} = \frac{31}{24} .$$

The general case for the sum of any two rational numbers is treated in a similar fashion. Let $\frac{a}{b}$ and $\frac{c}{d}$ name two rational numbers. These rational numbers are the solutions of the equations $bx = a$ and $dy = c$, respectively. By this is meant $b \cdot \frac{a}{b} = a$ and $d \cdot \frac{c}{d} = c$. Equivalently we have

$$d(b \cdot \frac{a}{b}) = da \qquad \text{and} \qquad b(d \cdot \frac{c}{d}) = bc.$$

Assuming the associative property, we can combine to get

$$(bd)\frac{a}{b} + (bd)\frac{c}{d} = ad + bc.$$

Note that the commutative property for the multiplication of whole numbers has been used to write db as bd and da as ad.

Using the distributive property gives

$$bd(\frac{a}{b} + \frac{c}{d}) = ad + bc.$$

Thus, if the operation of addition is to be extended in a natural way to rational numbers, we would want to say that $\frac{a}{b} + \frac{c}{d}$ is a solution of the equation

$$(bd)z = ad + bc.$$

This would lead us to conclude that

$$\frac{a}{b} + \frac{c}{d} \quad \text{and} \quad \frac{ad + bc}{bd}$$

name the same number.

Our thinking has led us to a plausible meaning for $\frac{a}{b} + \frac{c}{d}$. We have not offered a proof but rather an extended development to motivate a definition. Having arrived at this point, we can now wipe the slate clean and begin with the following:

Definition: The sum of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad + bc}{bd}$.

We have given a lengthy introduction to a relatively simple definition. There are several reasons for this verbosity. By doing this very slowly we hoped to convince the reader that addition of rational numbers is the work of man and that the definition was not deliberately designed to be as difficult as possible. The definition was arrived at through a review of the meaning of rational number and a desire to create binary operations which have properties we have found useful when working with addition and multiplication of whole numbers. It remains, of course, to be shown that this binary operation so defined does have these familiar properties, such as commutativity and associativity. In the next section we shall study the properties.

Many texts including Mathematics for Junior High School, Vol. 1, suggest that we add $\frac{2}{3}$ and $\frac{5}{8}$ by finding a common denominator. That is, one would write:

$$\begin{aligned} \frac{2}{3} + \frac{5}{8} &= \frac{2}{3} \cdot \frac{8}{8} + \frac{5}{8} \cdot \frac{3}{3} \\ &= \frac{16}{24} + \frac{15}{24} \\ &= \frac{1}{24}(16 + 15) \\ &= \frac{1}{24}(31) \\ &= \frac{31}{24}. \end{aligned}$$

The treatment in this text appears to be greatly different. However, the difference is more a philosophical difference than a mechanical difference. It will be seen that the mechanics of the two methods are really the same. The treatment in this text has been different to emphasize to the teacher that addition of rational numbers may be motivated and finally accomplished without multiplying fractions. The discussion of addition has depended upon the whole numbers. That is, we have made addition of rationals relate to addition and multiplication of whole numbers.

To see the similarity of the two methods, let us review the method of this text. To collect together $\frac{2}{3}$ and $\frac{5}{8}$ we multiply the equations

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 8 \cdot \frac{5}{8} = 5$$

by 8 and 3, respectively. This gives us

$$8 \cdot (3 \cdot \frac{2}{3}) = 8 \cdot 2 \quad \text{and} \quad 3 \cdot (8 \cdot \frac{5}{8}) = 3 \cdot 5.$$

The first equation is of the form $8 \cdot 3x = 8 \cdot 2$ which has a solution named by $\frac{8 \cdot 2}{8 \cdot 3}$. Hence, $\frac{8 \cdot 2}{8 \cdot 3} = \frac{2}{3}$. The second equation is of the form $3 \cdot 8x = 3 \cdot 5$ which has a solution named by $\frac{3 \cdot 5}{3 \cdot 8}$. Hence, $\frac{3 \cdot 5}{3 \cdot 8} = \frac{5}{8}$. We see that we have done the same work in both methods. Only the style is different.

Seventh grade texts generally introduce multiplication before addition, the reasons being that multiplication seems simpler than addition and that multiplication may be used in the computation of sums. Note, however, that both treatments use properties either to motivate the discussion or to carry out the computation.

Example: Use the definition of the sum of two rational numbers to find the sum $\frac{5}{18} + \frac{3}{8}$.

From the definition we have

$$\frac{5}{18} + \frac{3}{8} = \frac{5 \cdot 8 + 18 \cdot 3}{18 \cdot 8}$$

which may be written as

$$\frac{40 + 54}{144} = \frac{94}{144}.$$

The fraction $\frac{94}{144}$ names a rational number which has many names. At this stage we will not want to find the "simplest" name.

Class Exercises

1. Use the definition to find the sums:

   (a) $\frac{2}{3} + \frac{3}{5}$ (c) $\frac{1}{2} + \frac{1}{2}$ (e) $\frac{1}{3} + \frac{1}{6}$

   (b) $\frac{5}{2} + \frac{1}{8}$ (d) $\frac{1}{3} + \frac{1}{3}$ (f) $\frac{2}{4} + \frac{1}{2}$

2. The form of the answers to (c) and (d) of problem 1 will not be the same as the form of the sums obtained in the text. Are the sums themselves different?

3. (a) Use the definition to find the sum $\frac{4}{1} + \frac{3}{1}$.

   (b) Does the answer to part (a) agree with the fact that $\frac{4}{1}$ and $\frac{3}{1}$ are fractional names for 4 and 3?

4. (a) Use the definition to find the sum $\frac{28}{7} + \frac{10}{2}$.

   (b) Does the answer to part (a) agree with the fact that $\frac{28}{7}$ and $\frac{10}{2}$ are fractional names for 4 and 5, respectively.

6.2 Properties of Addition

The binary operation of addition on the rational numbers has been introduced and defined. Now is the time to investigate this operation to show that it does have the desired properties similar to addition on the whole numbers. We repeat that the sum of $\frac{a}{b}$ and $\frac{c}{d}$, $b, d, \neq 0$, is defined to be $\frac{ad + bc}{bd}$; the sum of $\frac{a}{b}$ and $\frac{c}{d}$ is the solution of the equation $bdx = ad + bc$. Since a, b, c, and d are whole numbers, so are bd and $ad + bc$. Thus, $\frac{a}{b} + \frac{c}{d}$ is the solution of an equation stated with whole numbers, $bd \neq 0$, which means $\frac{a}{b} + \frac{c}{d}$ is the name of a rational number. We have proved that the binary operation introduced in the last section is closed; the sum of two rational numbers is a rational number.

What else can we say about this binary operation? Let us compare $\frac{3}{4} + \frac{9}{10}$ and $\frac{9}{10} + \frac{3}{4}$. By the definition of addition, $\frac{3}{4} + \frac{9}{10}$ is

$$\frac{3 \cdot 10 + 4 \cdot 9}{4 \cdot 10}$$

and is the solution of the equation $4 \cdot 10x = 3 \cdot 10 + 4 \cdot 9$. By the definition of addition, $\frac{9}{10} + \frac{3}{4}$ is

$$\frac{9 \cdot 4 + 10 \cdot 3}{10 \cdot 4}$$

and is the solution of the equation $10 \cdot 4x = 9 \cdot 4 + 10 \cdot 3$. The equations are stated in terms of whole numbers as are the results of the definition. However, for the whole numbers multiplication and addition are commutative. Hence, we can show that the second result equals the first:

$$\frac{9 \cdot 4 + 10 \cdot 3}{10 \cdot 4} = \frac{4 \cdot 9 + 3 \cdot 10}{4 \cdot 10}$$ (commutative property for multiplication of whole numbers)

$$= \frac{3 \cdot 10 + 4 \cdot 9}{4 \cdot 10}.$$ (commutative property for addition of whole numbers)

Likewise, using the same properties we can show that the two equations are identical. We therefore conclude that

$$\frac{3}{4} + \frac{9}{10} = \frac{9}{10} + \frac{3}{4}.$$

The method that was used here will work in general to show that addition of rational numbers is commutative. For the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}.$$

That we can prove the commutative property holds for rational numbers is a reflection of the fact that the treatment of rationals in this text is "deeper" than that given in a junior high school text. We repeat once again: We don't expect the teacher to present this development to her classes but the teacher should see an orderly development of rational numbers.

The associative property may also be seen to hold for addition of rational numbers by following the definition through in much the same way as was done with the commutative property. In this case, however, the proof rests upon the use of the associative property for addition of whole numbers. When stated in terms of rational numbers $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, the associative property becomes

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right).$$

We have decided that the fractions such as $\frac{0}{1}$, $\frac{0}{2}$, and $\frac{0}{17}$, are different names for 0. Let us see how 0 behaves with respect to addition as we have defined this operation. To find the sum of $\frac{0}{1}$ and $\frac{3}{17}$ we use the definition:

$$\frac{0}{1} + \frac{3}{17} = \frac{0 \cdot 17 + 1 \cdot 3}{1 \cdot 17} = \frac{3}{17}.$$

That is, the rational number $\frac{0}{1}$ acts as an additive identity. (We have examined the behavior of $\frac{0}{1}$ only when combined with $\frac{3}{17}$ but it is clear, is it not, that the pattern would be the same with any rational number.)

Again we have not proved a general statement. Rather an indication has been given that 0 is an additive identity. For any rational number $\frac{a}{b}$, $b \neq 0$, it is true that $0 + \frac{a}{b} = \frac{a}{b}$. This should be interpreted as being true regardless of which name we use for 0. The reader should try a few examples. Is it true that $\frac{0}{1} + \frac{9}{11} = \frac{9}{11}$? Do the two sides of the equation $\frac{0}{23} + \frac{9}{11} = \frac{9}{11}$ name the same number?

We have seen that by carefully inventing the rational numbers and a binary operation on them we have a mathematical system with properties that are familiar to us.

Before leaving addition there is another matter which needs comment. It has been agreed that we will name a whole number with certain fraction names. Since 4 and $\frac{4}{1}$ solve the same equation we have agreed that 4 and $\frac{4}{1}$ name the same number. An addition for numbers with fraction names has just been described. Thus, given two whole numbers, say 4 and 3, we have two ways to perform addition. We may write $4 + 3 = 7$ or we may do the addition using fractions. The fractions $\frac{4}{1}$ and $\frac{3}{1}$ name the same numbers as 4 and 3:

$$\frac{4}{1} + \frac{3}{1} = \frac{4 \cdot 1 + 1 \cdot 3}{1 \cdot 1} = \frac{7}{1} .$$

Fortunately, $\frac{7}{1}$ and 7 name the same number. A grain of sand does not make a mountain, nor does one example prove a general statement. In this situation the one example does, however, give an insight into the general case.

The one example and the general statement which may be proved similarly tells us that the addition introduced on the rational numbers is an extension of the addition we know for whole numbers. This is highly desirable. We have two ways to add whole numbers; one is essentially finger counting, and the other is to rename the whole numbers as fractions and use addition of rational numbers. Had we obtained different answers for $4 + 3$ and $\frac{4}{1} + \frac{3}{1}$ our intuitive concept could not hold.

Class Exercises

5. Use the definition to perform the additions in (a), (b), (c), and (d).

   (a) $\frac{3}{4} + \frac{17}{29}$ (c) $\frac{0}{101} + \frac{6}{5}$

   (b) $\frac{17}{29} + \frac{3}{4}$ (d) $\frac{6}{5} + \frac{0}{101}$

   (e) Compare the answers to (a) and (b).

   (f) Compare the answers to (c) and (d).

6. (a) Express as a fraction the sum $\frac{^-3}{4} + \frac{2}{3}$.

   (b) Use the answer to (a) to put $(\frac{3}{4} + \frac{2}{3}) + \frac{6}{10}$ in fractional form.

   (c) Put $\frac{2}{3} + \frac{6}{10}$ in fractional form.

   (d) Use the answer to (c) to put $\frac{3}{4} + (\frac{2}{3} + \frac{6}{10})$ in fractional form.

   (e) Compare the answers to (b) and (d).

7. Put $\frac{18}{3} + \frac{25}{5}$ in fractional form using the definition of addition. Do the corresponding addition using non fractional names for these rational numbers. Compare your answers.

8. Students like to think that addition of $\frac{a}{b}$ and $\frac{c}{d}$ should be defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d} .$$

Why is this a most unsatisfactory definition?

Hint: Use the definition to find the sum $\frac{1}{2} + \frac{1}{2}$.

What should a teacher teach to a seventh grade class about rational numbers? Certainly, that our definition for the addition of rationals is reasonable. That is, the operation we call addition is defined exactly as our reason tells us it should be. Students tend to view rational numbers and operations on them as mysterious. This is particularly true for addition. The judicious use of equations can dispel much of this mystery. The properties of addition and of 0 should be stressed. The notion that 4 and $\frac{8}{2}$ must be identified as two names for the same number is probably too subtle at this point. However, the student will be willing to accept, without any discussion, that 4 and $\frac{8}{2}$ are two names for the same number.

A teacher who is confident working with rational numbers will be able to instill this confidence to the class. Operating on the rational numbers is not difficult. As has been seen, it depends only on a good working knowledge of the whole numbers. Class Exercise 8 may be used to discourage one prevalent false idea, particularly if illustrated with half dollars.

Students generally regard multiplication of rational numbers as simpler than addition. There are probably several reasons for this. To begin with, addition is introduced through the use of multiplication. Secondly, as teacher, we generally want our students to be efficient and use the lowest common denominator when adding. We frequently mark an answer wrong simply because it is not reduced. To illustrate this, take the problem of putting $\frac{5}{36} + \frac{7}{30}$ in fractional form. Following our method the sum would be $\frac{402}{1080}$. We motivated this by multiplying the equation $36x = 5$ and $30y = 7$ by 30 and 36, respectively, to obtain in each case the coefficient 1080. The common coefficient suggests the distributive law. We could also have obtained a common coefficient of 180 by multiplying the equations by 5 and 6, respectively, to obtain $5 \cdot 36x = 5 \cdot 5$ and $6 \cdot 30y = 6 \cdot 7$ or $180x = 25$ and $180y = 42$; thus, $180(x + y) = 67$. It is true that our second answer appears simpler and that we prefer the answer $\frac{67}{180}$. This is not mathematical reasoning but psychological. We must remember that it is better to get a correct answer rather than to worry too much about efficiency.

The essence of addition as usually taught in the elementary grades is finding a common denominator. When we insist, at an early stage, that the student find the least common denominator, the student may lose sight of the meaning of addition. The student should thoroughly learn that rational numbers have many names. The most useful name will depend on the circumstances.

6.3 Multiplication

Now that addition of rational numbers has been introduced, we wish to introduce a second binary operation. The operation of addition was motivated through the properties and operations of the whole numbers. The second binary operation, multiplication, will also be motivated through the whole numbers.

Let us look first at an example. The rational number $\frac{3}{4}$ is the solution of the equation $4x = 3$; $4 \cdot \frac{3}{4} = 3$. Also, $\frac{7}{5}$ is the solution of the equation $5y = 7$; $5 \cdot \frac{7}{5} = 7$. The numbers 3 and 7 have a well determined product, $3 \cdot 7 = 21$. As $4 \cdot \frac{3}{4}$ and $5 \cdot \frac{7}{5}$ are other names for 3 and 7

they, too, have a well determined product. Thus, it is meaningful to write

$$(4 \cdot \frac{3}{4}) \cdot (5 \cdot \frac{7}{5}) = 3 \cdot 7 .$$

Remember we are merely exploring, not proving, and so may use a bit of sleight of hand to rewrite this equation as

$$(4 \cdot 5) \cdot (\frac{3}{4} \cdot \frac{7}{5}) = 3 \cdot 7.$$

(Here we have proceeded as if it is meaningful to use associativity and commutativity for the operation of multiplication with rational numbers.)

The displayed equation above does suggest to us that $\frac{3}{4} \cdot \frac{7}{5}$ should be the solution of

$$(4 \cdot 5)z = 3 \cdot 7.$$

The solution of this equation is named $\frac{3 \cdot 7}{4 \cdot 5}$. Hence, it seems reasonable to say

$$\frac{3}{4} \cdot \frac{7}{5} = \frac{3 \cdot 7}{4 \cdot 5} = \frac{21}{20} .$$

Definition: The product of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ac}{bd}$.

This definition may be rephrased as: The product of two rational numbers written as fractions may be represented as a fraction whose numerator is the product of the numerators, and whose denominator is the product of the denominators.

Once again the reader has observed a slightly different approach to the treatment of rational numbers. The reason for this approach is as before, to emphasize the equation meaning of rational numbers. The rational numbers are known through equations, and the equations have been used to motivate the definitions. We also use the properties that we would like addition and multiplication to possess to suggest these operations to us. Having arrived at what seem to be reasonable ideas of addition and multiplication, we then show that the properties hold. Other treatments take the properties for granted and show that the definitions given must be the definitions used. The net result is, of course, the same operation.

The slight differences in introducing rational numbers are not as important as the similarity in the newer texts. This similarity is a pedagogical similarity. Rather than present the arithmetic of rationals as an irrevocable law of nature, which we must all unthinkingly obey, arithmetic is presented as an organized, conscious development of man to suit his purposes. Children sometimes ask questions about mathematics that seem naive but are really penetrating. "Who decided $1 + 1 = 2$?," or "Who decided that $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$?" These and other questions have answers when mathematics is developed rather than merely presented as a fact.

Class Exercises

9. Use the definition of multiplication to find the product $\frac{4}{1} \cdot \frac{3}{1}$. Does the answer agree with the fact that $\frac{4}{1}$ and $\frac{3}{1}$ are fractional names for 4 and 3 ?

10. Use the definition to find the product $\frac{28}{7} \cdot \frac{10}{2}$. Does this answer agree with the fact that $\frac{28}{7}$ and $\frac{10}{2}$ are fractional names for counting numbers?

6.4 Properties of Multiplication

We would like to show that multiplication of rational numbers, as defined, is as well behaved as addition. That is, we would like to show that multiplication is closed, commutative, and associative. Furthermore, we like to know that there is an identity with respect to multiplication and that multiplication of rationals is an extension of multiplication of whole numbers.

For whole numbers $a, b, c,$ and d with $b, d \neq 0$, the product of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{ac}{bd}$. The symbol $\frac{ac}{bd}$ names the solution of the equation $(bd)x = ac$. Thus, the product of two rational numbers is again a rational number. The binary operation of multiplication is closed.

Do not expect a seventh grader to turn cartwheels in the aisle as you announce this fact. His reaction is apt to be the bored "So what," or the pseudo sophisticated "Of course." It may well be beneficial to repeat some examples of binary operations which are not closed: Subtraction on the set of counting numbers, division on the set of whole numbers, or multiplication on the set of numbers $1, 2, 3, 4$.

Is multiplication commutative? For rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ this asks: Is it true that $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$? We have $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{c}{d} \cdot \frac{a}{b} = \frac{ca}{db}$. Do the two fractions $\frac{ac}{bd}$ and $\frac{ca}{db}$ name the same number? Using the commutative property for multiplication of whole numbers the numerator and denominator of the second fraction can be shown to equal those of the first. Hence, the multiplication of rational numbers is commutative.

Given three rational numbers it is not difficult to show that the associative property holds for multiplication. Indeed, it is no more difficult to show the associative property in general than in a special case. Some examples will convince the reader of this.

The number 1 has a special property with respect to multiplication of whole numbers. Does this property extend to the rational numbers? Let us look at an example. To find the product of 1 and $\frac{4}{7}$ we must first rename 1 with a fractional name. There are many names to choose from; suppose we use $\frac{3}{3}$. The product $\frac{3}{3} \cdot \frac{4}{7} = \frac{3 \cdot 4}{3 \cdot 7} = \frac{12}{21}$. The number named by $\frac{12}{21}$ is the same number as named by $\frac{4}{7}$.

To show that 1 is a multiplicative identity we use the name $\frac{k}{k}$, $k \neq 0$, for 1 and take an arbitrary rational number $\frac{a}{b}$. Now $\frac{k}{k} \cdot \frac{a}{b} = \frac{ka}{kb}$. In Class Exercise 15 of Chapter 5 it was shown that $\frac{ka}{kb}$ and $\frac{a}{b}$ name the same number. We frequently write this as $1 \cdot \frac{a}{b} = \frac{a}{b}$. Thus, we have proved that 1 is the multiplicative identity.

The whole numbers have been identified with certain rational numbers. When we say that the whole numbers are a subset of the rational numbers, we mean that there is a subset of the rational numbers that solve the same equations as do the whole numbers. Now an operation called multiplication has been introduced on the rational numbers. Seemingly then, there are two forms of multiplication for whole numbers: The multiplication as learned for whole numbers, and the multiplication forced on the whole numbers when they are regarded as a subset of the rationals. As an example there is the product $9 \cdot 7 = 63$. We also may write 9 as $\frac{18}{2}$ and 7 as $\frac{35}{5}$, then the product $\frac{18}{2} \cdot \frac{35}{5}$ equals $\frac{630}{10}$, which is easily seen to be another name for $\frac{63}{1}$ or to be identified with 63. This is true in general; the product of two whole numbers determined with fractions or with decimal numerals is the same. This is much like two students who work the same arithmetic problem. If one student uses blue ink and another uses black, the external form will be different but the arithmetical result will be the same.

Class Exercises

11. Show by computing that

$$\frac{3}{4} \cdot (\frac{9}{7} \cdot \frac{8}{11}) = (\frac{3}{4} \cdot \frac{9}{7}) \cdot \frac{8}{11} .$$

12. Show by computation that

$$\frac{3}{4} \cdot (\frac{9}{7} + \frac{8}{11}) = (\frac{3}{4} \cdot \frac{9}{7}) + (\frac{3}{4} \cdot \frac{8}{11}).$$

13. Determine the rational number which is a solution of the equation

$$\frac{3}{8} \cdot x = \frac{7}{5} .$$

14. Evaluate:

(a) $\frac{3}{4} \cdot \frac{4}{3}$ (c) $\frac{2}{5} \cdot \frac{5}{2}$

(b) $\frac{9}{10} \cdot \frac{10}{9}$ (d) $\frac{9}{7} \cdot \frac{7}{9}$

15. Is it meaningful to write $1 = \frac{96}{96}$? If so, in what respect is it meaningful?

6.5 The Distributive Property

Problem 12 of the preceding set of Class Exercises is an example of the working of the distributive property. For rational numbers the distributive property states:

> For a, b, c, d, e, and f whole numbers with b, d, f, $\neq 0$, it is true that
>
> $$\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} \cdot \frac{c}{d}) + (\frac{a}{b} \cdot \frac{e}{f}).$$

It is a straightforward approach to show that the property holds. Simply compute both sides and see that they name the same rational number. The left side can be expressed as follows:

$$\begin{aligned}\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) &= \frac{a}{b} \cdot (\frac{cf + de}{df}) \\ &= \frac{a(cf + de)}{b(df)} \\ &= \frac{a(cf) + a(de)}{b(df)} .\end{aligned}$$

The last step uses the distributive property for whole numbers. The right side can be expressed as follows:

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) + \left(\frac{a}{b} \cdot \frac{e}{f}\right) = \frac{ac}{bd} + \frac{ae}{bf}$$

$$= \frac{(ac)\cdot(bf)+(bd)\cdot(ae)}{(bd)\cdot(bf)} .$$

The two fractions $\frac{a(cf) + a(de)}{b(df)}$ and $\frac{(ac)\cdot(bf)+(bd)\cdot(ae)}{(bd)\cdot(bf)}$ are quite different in appearance. Do they name the same number? Using the properties of whole numbers the two fractions may be rewritten as

$$\frac{acf + ade}{bdf} \quad \text{and} \quad \frac{b(acf + ade)}{b(bdf)} .$$

From Class Exercise 15 of Chapter 5 it follows that the two fractions name the same rational number. Thus, it has been shown that the distributive law is valid.

6.6 Subtraction

With the introduction of addition and multiplication it is possible to introduce subtraction and division. Subtraction and division are not to be regarded as new operations. It shall be shown that addition and multiplication can be used to solve problems that are usually considered as subtraction and division problems.

We may determine by the operation addition a rational number we call

$$\frac{3}{4} + \frac{7}{5} .$$

To determine the rational number we call $\frac{2}{3} - \frac{4}{7}$, we may proceed in a routine way:

$$\frac{2}{3} - \frac{4}{7} = \frac{2}{3} \cdot \frac{7}{7} - \frac{4}{7} \cdot \frac{3}{3}$$

$$= \frac{14}{21} - \frac{12}{21}$$

$$= \frac{1}{21}(14 - 12)$$

$$= \frac{1}{21} \cdot 2$$

$$= \frac{2}{21} .$$

(In the third step we have made use of a distributive property of multiplication over subtraction.)

Let us look deeper into the meaning of the rational number we call $\frac{2}{3} - \frac{4}{7}$ by use of the equation method. When we evaluate $\frac{2}{3} - \frac{4}{7}$, we determine a value for x such that $\frac{4}{7} + x = \frac{2}{3}$. This is comparable to saying that the solution of the equation $9 + n = 15$ is $15 - 9$. To see more clearly the relation of subtraction to addition let us use the equation $\frac{4}{7} + x = \frac{2}{3}$. to find $\frac{2}{3} - \frac{4}{7}$.

When we attempt to solve this equation we are asking: Is there a rational number which may be substituted for x in

$$\frac{4}{7} + x = \frac{2}{3}$$

to make a true statement? To have something to talk about, let us think of x as a rational number, $\frac{u}{v}$. We wish to determine whole number replacements for u and v such that

$$\frac{4}{7} + \frac{u}{v} = \frac{2}{3} .$$

If there are such numbers, we may add $\frac{4}{7}$ and $\frac{u}{v}$ to obtain $\frac{4v + 7u}{7v} = \frac{2}{3}$. These two fractions, $\frac{4v + 7u}{7v}$ and $\frac{2}{3}$, will name the same number if

$$3(4v + 7u) = 2 \cdot 7v ,$$

by the definition of equivalent fractions. To see if it is possible to determine values for u and v so that this equation holds, let us rewrite it:

$$\begin{aligned} 3(4v + 7u) &= 2 \cdot 7v \\ 12v + 21u &= 14v \\ 21u &= 14v - 12v \end{aligned}$$

which by a distributive property becomes

$$\begin{aligned} 21u &= (14 - 12)v \\ 21u &= 2v . \end{aligned}$$

It now seems clear that we should try $v = 21$ and $u = 2$. At least these values for u and v will make the statement $21u = 2v$ a true statement. Thus, it has been suggested that $x = \frac{2}{21}$. Let us see if this works: The sum $\frac{4}{7} + \frac{2}{21}$ is $\frac{98}{147}$ which can easily be shown to name the same number as $\frac{2}{3}$.

This technique for subtraction depends only on a knowledge of addition and the meaning of fractions. A more routine technique for solving such problems may readily be presented. This method depends upon the statement: For any counting number k, the fractions $\frac{a}{b}$ and $\frac{ak}{bk}$ name the same number.

To determine $\frac{u}{v}$ such that

$$\frac{4}{7} + \frac{u}{v} = \frac{2}{3}$$

we first rename $\frac{4}{7}$ and $\frac{2}{3}$ so that they have the same denominator. Since $7 \cdot 3 = 3 \cdot 7$ we write $\frac{4}{7} = \frac{4 \cdot 3}{7 \cdot 3}$ and $\frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7}$. The equation we wish to solve may be written as

$$\frac{12}{21} + \frac{u}{v} = \frac{14}{21} .$$

It is clear that equality will hold if $\frac{u}{v}$ is $\frac{2}{21}$. This latter method, determining $\frac{u}{v}$ such that $\frac{12}{21} + \frac{u}{v} = \frac{14}{21}$, is really subtraction as taught in seventh grade. We wish to find $\frac{2}{3} - \frac{4}{7}$. The fractions $\frac{2}{3}$ and $\frac{4}{7}$ are renamed as:

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{7}{7} = \frac{14}{21}$$

and

$$\frac{4}{7} = \frac{4}{7} \cdot \frac{3}{3} = \frac{12}{21} .$$

Hence,

$$\frac{2}{3} - \frac{4}{7} = \frac{14}{21} - \frac{12}{21} = \frac{2}{21} .$$

Class Exercises

16. Solve each of the equations by the two methods given above.

    (a) $\frac{9}{17} + x = \frac{5}{4}$

    (b) $\frac{2}{3} + x = \frac{4}{5}$

    (c) $\frac{9}{17} + x = \frac{9}{17}$

    (d) $\frac{99}{23} + y = \frac{99}{23}$

17. Is there any similarity between the two methods of subtraction described in this section?

The second method given above is undoubtedly the preferred method to use in the seventh grade. The first method has the advantage of stressing our basic information about rationals.

When the whole numbers are used to construct the non-negative rational numbers, we may easily write equations that do not have solutions in the set of non-negative rationals. For example, consider the equation:

$$\frac{2}{3} + x = \frac{4}{7} .$$

There is a rational number solution to this equation but it is the negative rational number, $\frac{-2}{21}$. Problems of this type can be solved using the methods previously described. However, they require familiarity with the fundamental operations on integers. These will be summarized in the last section of this chapter. We should, however, keep in mind that the procedures illustrated thus far with non-negative rational numbers may be easily extended to include all rational numbers, positive, negative, and zero.

6.7 Division

Division of whole numbers was introduced through its relationship to the multiplication of whole numbers. Corresponding to the multiplication $5 \cdot 2 = 10$ we have the divisions $10 \div 2 = 5$ and $10 \div 5 = 2$. Corresponding to the multiplication $3 \cdot n = 15$ we have the divisions $15 \div n = 3$ and $15 \div 3 = n$. Hence, to find the value of n such that $3 \cdot n = 15$ we may name n by $15 \div 3$ or $\frac{15}{3}$.

We shall also use multiplication as the basic operation in introducing division of rational numbers. To solve an equation of the form $\frac{a}{b} \cdot x = \frac{c}{d}$, we note that the result x can be expressed as $\frac{c}{d} \Big/ \frac{a}{b}$ or as $\frac{c}{d} \div \frac{a}{b}$. How is this result to be evaluated and is it a rational number?

To help answer these questions let us begin the discussion with a numerical example, say

$$\frac{3}{4} \cdot x = \frac{7}{5} .$$

As we hope to find a rational number to replace x let us think of x as $\frac{u}{v}$. We want to determine the whole number replacements for u and v, $v \neq 0$, in such a way that

$$\frac{3}{4} \cdot \frac{u}{v} = \frac{7}{5} .$$

By use of the definition of multiplication this equation may be written as

$$\frac{3u}{4v} = \frac{7}{5} .$$

Here we wish to determine u and v such that $\frac{3u}{4v}$ and $\frac{7}{5}$ name the same number since this is what the last equation means. By the criterion agreed upon in Chapter 5, $\frac{a}{b}$ and $\frac{c}{d}$ will name the same number if and only if $ad = bc$. Hence, for our example, $\frac{3u}{4v}$ and $\frac{7}{5}$ will name the same number if and only if

$$(3u) \cdot 5 = (4v) \cdot 7 .$$

Using the commutative property this can be expressed as

$$5 \cdot (3u) = 7 \cdot (4v) .$$

Can u and v be determined so that this last equation will be a true statement? There are many ways this can be done. The simplest way is to observe that on the left 3 and 5 appear as factors. To balance the equation let us make 3 and 5 appear as factors on the right side. This may be accomplished by choosing $v = 3 \cdot 5$.

$$5 \cdot (3u) = (7 \cdot 4) \cdot (3 \cdot 5).$$

Now to choose u so that equality holds, we assign u the value $7 \cdot 4$.

While other values of v and u would also have made the original equation true, we have chosen $v = 3 \cdot 5$ and $u = 7 \cdot 4$. That is, we have decided to choose x such that

$$x = \frac{u}{v} = \frac{7 \cdot 4}{3 \cdot 5} .$$

Let us go back to $\frac{3}{4} \cdot x = \frac{7}{5}$ and see if this replacement for x works as expected. Is it true that

$$\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5} = \frac{7}{5} ?$$

Do $\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5}$ and $\frac{7}{5}$ name the same number?

As $\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5}$ may be written as $\frac{3 \cdot (7 \cdot 4)}{4 \cdot (3 \cdot 5)}$ we may conclude that the two are equal and that the equation has been solved.

Since $\frac{7 \cdot 4}{3 \cdot 5}$ is the replacement for x that will make $\frac{3}{4} \cdot x = \frac{7}{5}$ a true statement and since we want $\frac{7}{5} \div \frac{3}{4}$ as a solution, it follows that

$$\frac{7}{5} \div \frac{3}{4} = \frac{7 \cdot 4}{3 \cdot 5} .$$

Using the commutative property for multiplication of whole numbers and the definition of multiplication for rational numbers, we get

$$\frac{7}{5} \div \frac{3}{4} = \frac{7}{5} \cdot \frac{4}{3} .$$

This illustrates the common rule used for the division of rational numbers when expressed as fractions.

This method works just as easily in all cases. Let us examine the general case. To solve the equation

$$\frac{a}{b} \cdot x = \frac{c}{d}$$

we think of x as a rational number, say $\frac{u}{v}$. Replacing x with the symbol

$\frac{u}{v}$ and multiplying yields

$$\frac{a \cdot u}{b \cdot v} = \frac{c}{d} .$$

For these two fractions, $\frac{a \cdot u}{b \cdot v}$ and $\frac{c}{d}$, to name the same number, we must have, if possible,

$$(a \cdot u) \cdot d = (b \cdot v) \cdot c.$$

Using the associative and commutative properties gives

$$(a \cdot d) \cdot u = (b \cdot c) \cdot v.$$

Following the method of the special case, we may choose u to be the whole number $(b \cdot c)$ and v to be $(a \cdot d)$. Thus, $\frac{u}{v} = \frac{b \cdot c}{a \cdot d}$.

Is it true that

$$\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d} \ ?$$

Before you answer this leading question, recall that rational numbers in fractional form have non-zero denominators. This means that $a \cdot d$ must not be zero. As it was implied that $\frac{c}{d}$ is a rational number, $d \neq 0$. We must, therefore, also require that $a \neq 0$. With this restriction, $a \neq 0$, we get the following development:

$$\begin{aligned} \frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} &= \frac{a \cdot (b \cdot c)}{b \cdot (a \cdot d)} \\ &= \frac{(a \cdot b) \cdot c}{(b \cdot a) \cdot d} \\ &= \frac{a \cdot b}{b \cdot a} \cdot \frac{c}{d} \\ &= \frac{a \cdot b}{a \cdot b} \cdot \frac{c}{d} \\ &= \frac{a}{a} \cdot \frac{b}{b} \cdot \frac{c}{d} \\ &= 1 \cdot 1 \cdot \frac{c}{d} \\ &= \frac{c}{d} . \end{aligned}$$

Thus, $\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d}$.

Just as $5 \cdot 2 = 10$ conveys the same information as $10 \div 5 = 2$, so does $\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d}$ convey the same information as

$$\frac{c}{d} \div \frac{a}{b} = \frac{b \cdot c}{a \cdot d} .$$

The result $\frac{b \cdot c}{a \cdot d}$ is a rational number. The last equation can, of course, be expressed as

$$\frac{c}{d} \div \frac{a}{b} = \frac{c}{d} \cdot \frac{b}{a} .$$

What has been shown to be true is the well-known rule: To divide one rational number by another when both are expressed in fractional form, invert the divisor and multiply.

Again our development is based upon the use of whole numbers in forming non-negative rational numbers. The procedure for division can, however, be extended to include negative numbers.

Class Exercises

18. Solve each of the following using an analysis similar to that in the text.

(a) $\frac{2}{3} \cdot x = \frac{7}{10}$ (d) $\frac{17}{23} \cdot x = \frac{17}{23}$

(b) $\frac{19}{20} \cdot x = \frac{43}{31}$ (e) $\frac{2}{3} \cdot y = \frac{3}{2}$

(c) $\frac{2}{3} \cdot x = \frac{4}{3}$

19. Which of the following equations have unique solutions?

(a) $\frac{0}{3} \cdot x = \frac{2}{3}$ (d) $\frac{0}{17} \cdot x = \frac{0}{7}$

(b) $\frac{4}{3} \cdot x = \frac{0}{7}$ (e) $4 \cdot x = \frac{7}{9}$

(c) $\frac{2}{5} \cdot x = \frac{2}{5}$ (f) $\frac{5}{8} \cdot x = 4$

6.8 Operations on the Integers

Thus far we have restricted our discussion of the operations on rationals to non-negative rational numbers. The reason for this is that most seventh grade students will study computations using the numbers of arithmetic (the non-negative rationals) well before they meet the set of integers or the negative rationals. The teacher, however, should see that the definitions and rules thus far established necessarily must apply to all rational numbers, positive, zero, and negative. The extension is easy once the operating rules for the integers are established.

Recall that in Chapter 5 the set of integers was shown to contain the set of counting numbers, zero, and the set of opposites of the counting numbers. These subsets of the set of integers were called the positive integers, zero, and the negative integers, respectively. In symbols we can represent the set of integers as:

$$I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\} .$$

The identity element for addition using the set of integers is 0 since for every integer $\underline{a}$, $a + 0 = 0 + a = a$. In the set of integers each element also has an opposite called its additive inverse. In Chapter 4 two elements were defined as inverses of each other under a given binary operation if the result of this operation on the two elements is the identity element for that operation. Thus, additive inverses for the integers are integers which when added give the identity element, 0. Note in the examples how the opposites serve as additive inverses.

$$2 + {}^{-}2 = 0$$
$${}^{-}5 + 5 = 0$$
$$0 + 0 = 0$$

The operation of addition with integers was introduced in the last chapter using the number line. Subtraction can be handled in much the same way by making use of the fact that if a and b are integers, then $a - b = a + {}^{-}b$. This property of subtraction allows us to change every subtraction problem into an addition problem. For example:

$$\begin{aligned} 7 - 11 &= 7 + ({}^{-}11) = {}^{-}4 \\ ({}^{-}7) - 11 &= ({}^{-}7) + ({}^{-}11) = {}^{-}18 \\ 7 - ({}^{-}11) &= 7 + 11 = 18 \\ ({}^{-}7) - ({}^{-}11) &= ({}^{-}7) + 11 = 4 \end{aligned}$$

In general, subtracting a number is equivalent to adding its additive inverse.

Consider next the multiplication of integers. We know from the properties of whole numbers that the product of two positive integers is a positive integer. We also know that the product of zero and any integer is zero. But how should we define the product of a positive and a negative integer?

The product $4 \cdot ({}^{-}7)$ may be expressed as the sum $({}^{-}7) + ({}^{-}7) + ({}^{-}7) + ({}^{-}7)$ which we know equals ${}^{-}28$. Also, since we want the commutative property to hold, we will agree that $4 \cdot ({}^{-}7)$ and $({}^{-}7) \cdot 4$ mean the same integer. That is,

$$4 \cdot ({}^{-}7) = ({}^{-}7) \cdot 4.$$

Another way to evaluate $4 \cdot ({}^{-}7)$ is illustrated here. From the property of additive inverses we know that

$$7 + ({}^{-}7) = 0.$$

Hence, we may write

$$4 \cdot [7 + (^-7)] = 4 \cdot 0 = 0.$$

Using the distributive property for integers, we then get

$$4 \cdot (7) + 4 \cdot (^-7) = 0.$$

Since $4 \cdot (7)$ and $4 \cdot (^-7)$ add to zero, they must be additive inverses or opposites. But $4 \cdot (7)$ is 28. Now since $4 \cdot (^-7)$ is the additive inverse of $4 \cdot (7)$, it must be the additive inverse of 28. Thus, we conclude that $4 \cdot (^-7) = {}^-28$.

The two methods shown give the same results. In general, we can say that the product of a positive integer and a negative integer is a negative integer.

What meaning should we give to $(^-4) \cdot (^-7)$? Proceeding as before we get the following:

$$7 + (^-7) = 0$$
$$(^-4) \cdot [7 + (^-7)] = (^-4) \cdot 0 = 0$$
$$(-4) \cdot 7 + (^-4) \cdot (^-7) = 0.$$

Since $(^-4) \cdot 7$ and $(^-4) \cdot (^-7)$ add to zero, they are additive inverses. But $(^-4) \cdot 7 = {}^-28$. Thus, we conclude that $(^-4) \cdot (^-7)$ is the additive inverse of ${}^-28$, or $(^-4) \cdot (^-7) = 28$. The same development will hold for any two negative integers. In general, we say that the product of two negative integers is a positive integer.

An interesting and informal introduction to the product of integers using patterns in a multiplication table is given in the SMSG publication, Mathematics for Junior High School, Vol. 2. The procedure for division of integers follows directly from the multiplication procedure. If two positive or two negative integers are divided, the quotient is positive. If a positive and a negative integer are divided, the quotient is negative.

Class Exercises

20. Do each problem using the fact that $a - b = a + (^-b)$.

    (a) $17 - 21$
    (b) $(^-17) - 21$
    (c) $17 - (^-21)$
    (d) $(^-17) - (^-21)$

21. Evaluate each product.

    (a) $7 \cdot 13$
    (b) $(^-7) \cdot 13$
    (c) $7 \cdot (^-13)$
    (d) $(^-7) \cdot (^-13)$

22. Evaluate each quotient.

(a) $81 \div 3$ (c) $81 \div ({}^{-}3)$

(b) $({}^{-}81) \div 3$ (d) $({}^{-}81) \div ({}^{-}3)$

Let us look again at the operations this time using negative rational numbers. In general, the definition of the four basic operations and the properties developed in this chapter for the non-negative rational numbers can be extended to include also the negative rational numbers. In so doing, however, we will make use of a modified definition of rational numbers. We may define the set of rational numbers as all numbers that can be expressed in the form $\frac{a}{b}$ where a is an integer and b is a counting number. This change in definition now admits the negative rational numbers. The corresponding change in the equation definition would be that the rational numbers is the set of all solutions of equations in the form $bx = a$ where a is an integer and b a counting number.

With this change we can now ask for solutions of equations such as

$$4x = {}^{-}3$$

and know that they will be rational numbers. The solution here is ${}^{-}(\frac{3}{4})$, the additive inverse of $\frac{3}{4}$. The solution may be written as $\frac{{}^{-}3}{4}$ with the numerator of the fraction a negative integer.

The properties previously established for the non-negative rational numbers will hold for the negative rationals as well. Likewise, the definitions of the four fundamental operations apply to all rational numbers through the properties of integers. In the following examples study how the operations involving negative rational numbers have been completed by making use of our knowledge of integers.

$$\frac{1}{2} + \frac{{}^{-}3}{4} = \frac{1 \cdot 4 + 2 \cdot ({}^{-}3)}{2 \cdot 4} = \frac{4 + ({}^{-}6)}{8} = \frac{{}^{-}1}{4}$$

$$\frac{{}^{-}2}{3} - \frac{3}{4} = \frac{{}^{-}2}{3} + \frac{{}^{-}3}{4} = \frac{({}^{-}2) \cdot 4 + 3 \cdot ({}^{-}3)}{4 \cdot 3} = \frac{({}^{-}8) + ({}^{-}9)}{12} = \frac{{}^{-}17}{12}$$

$$\frac{{}^{-}1}{5} \cdot \frac{{}^{-}3}{4} = \frac{({}^{-}1) \cdot ({}^{-}3)}{5 \cdot 4} = \frac{3}{20}$$

$$\frac{{}^{-}2}{5} \div \frac{3}{4} = \frac{{}^{-}2}{5} \cdot \frac{4}{3} = \frac{({}^{-}2) \cdot 4}{5 \cdot 3} = \frac{{}^{-}8}{15}$$

Chapter Exercises

1. Evaluate:

   (a) $\frac{7}{8} + \frac{11}{15}$ (c) $\frac{7}{8} \cdot \frac{11}{15}$

   (b) $\frac{7}{8} - \frac{11}{15}$ (d) $\frac{7}{8} \div \frac{11}{15}$

2. Find the following sums:

   (a) $1 + \frac{1}{2}$ (c) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

   (b) $1 + \frac{1}{2} + \frac{1}{4}$ (d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

3. Use the results of Exercise 2 to make an informal guess of the following sums:

   (a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{32} + \frac{1}{64}$

   (b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{512} + \frac{1}{1024}$

4. Find $(\frac{3}{4} \div \frac{9}{10}) \div \frac{1}{2}$ and $\frac{3}{4} \div (\frac{9}{10} \div \frac{1}{2})$. Are the answers the same? What conclusion can be drawn from the last answer?

5. Find $\frac{3}{4} \div (\frac{2}{3} - \frac{1}{2})$ and $(\frac{3}{4} \div \frac{2}{3}) - (\frac{3}{4} \div \frac{1}{2})$. Are the answers the same?

6. Using the rational numbers $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, prove the associative property for multiplication.

7. Evaluate:

   (a) $9 - 13$ (c) $(^-9) - 13$

   (b) $9 - (^-13)$ (d) $(^-9) - (^-13)$

8. Evaluate:

   (a) $(\frac{^-3}{4}) - \frac{3}{8}$ (c) $(\frac{^-1}{2}) \cdot (\frac{^-3}{4})$

   (b) $\frac{5}{7} + (\frac{^-3}{4})$ (d) $(\frac{^-2}{3}) \div \frac{5}{6}$

9. The sums $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ and $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$ are examples of fractions written as the sum of unit fractions, i.e., fractions with numerator 1. Represent each of the following rational numbers as sums of unit fractions. A particular unit fraction may be used only once in each sum.

   (a) $\frac{13}{12}$ (c) $\frac{9}{20}$

   (b) $\frac{8}{15}$ (d) $\frac{47}{60}$

10. (a) Show that $\frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}$.

(b) Show that $\frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$.

(c) Show that $\frac{1}{4 \cdot 5} = \frac{1}{4} - \frac{1}{5}$.

(d) Express $\frac{1}{9 \cdot 10}$ as the difference of two unit fractions.

(e) Express $\frac{1}{18 \cdot 19}$ as the difference of two unit fractions.

(f) Use the results of the above to find the sum
$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{18 \cdot 19}$.

11. Find the sums in (a), (b), and (c):

(a) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3}$

(b) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}$

(c) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}$

(d) Make an educated guess as to the sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{18 \cdot 19}$$

(e) Make an educated guess as to the sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{99 \cdot 100}.$$

12. Solve the equation

$$\frac{3}{5} \cdot x + \frac{3}{16} = \frac{2}{3} + \frac{1}{4}.$$

13. A "magic square" is a square array of numbers such that each row, each column, and the two diagonals all add to the same number. Complete the table below to form a 3 × 3 "magic square."

$\frac{2}{3}$		
$\frac{1}{4}$		
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$

Answers to Class Exercises

1. (a) $\frac{19}{15}$ (b) $\frac{42}{16}$ (c) $\frac{4}{4}$ (d) $\frac{6}{9}$ (e) $\frac{9}{18}$ (f) $\frac{8}{8}$

2. No, in the sense that we have different fractional representations of the same rational numbers.

$$\frac{4}{4} = 1 \quad \text{and} \quad \frac{6}{9} = \frac{2}{3}$$

3. (a) $\frac{7}{1}$ (b) Yes, the rational number $\frac{7}{1}$ is a name for the solution of the equation $1x = 7$ which also has a solution named 7. We have agreed therefore that $\frac{7}{1}$ and 7 name the same number.

4. (a) $\frac{126}{14}$. (b) Yes, the number $\frac{126}{14}$ is the solution of $14y = 126$ which also has a solution $y = 9$. Thus, following our agreement, 9 and $\frac{126}{14}$ are names for the same rational number.

5. (a) $\frac{152}{116}$ (b) $\frac{152}{116}$ (c) $\frac{606}{505}$ (d) $\frac{606}{505}$

   (e) They are the same, illustrating the commutative property.

   (f) They are the same. Since $\frac{606}{505} = \frac{6 \cdot 101}{5 \cdot 101} = \frac{6}{5}$ this illustrates the identity property of 0.

6. (a) $\frac{17}{12}$ (b) $\frac{242}{120}$ (c) $\frac{38}{30}$ (d) $\frac{242}{120}$

   (e) They are the same, illustrating the associative property.

7. $\frac{165}{15}$. $6 + 5 = 11$. The symbols 11 and $\frac{165}{15}$ both designate the answer to $15z = 165$.

8. If $\frac{1}{2} + \frac{1}{2}$ is computed this way, one obtains $\frac{2}{4}$ which would not satisfy one's intuition. Also, the distributive law would clearly fail to hold.

9. $\frac{12}{1}$. Both $\frac{12}{1}$ and $12 = 4 \cdot 3$ name the solution of $1x = 12$.

10. $\frac{280}{14}$. Yes, $\frac{28}{7} \cdot \frac{10}{2} = \frac{280}{14} = \frac{14 \cdot 20}{14 \cdot 1} = \frac{20}{1} = 20$. Also, $\frac{28}{7} = 4$, $\frac{10}{2} = 5$, and $4 \cdot 5 = 20$.

11. The result of both computations is $\frac{216}{308}$.

12. The left-hand side is $\frac{465}{308}$ and the right-hand side is $\frac{1860}{1232}$.
The two may be shown to be equal by computing $465 \cdot 1232$ and $1860 \cdot 308$ or by observing $\frac{1860}{1232} = \frac{4 \cdot 465}{4 \cdot 308} = \frac{465}{308}$.

13. $\frac{26}{15}$

14. (a) $\frac{12}{12}$ (b) $\frac{90}{90}$ (c) $\frac{10}{10}$ (d) $\frac{63}{63}$

What interesting fact is observed?

15. Yes, for both 1 and $\frac{96}{96}$ are names of the solution of $96w = 96$.

16. (a) $\frac{49}{88}$ (c) $\frac{0}{a}$, a any counting number

(b) $\frac{2}{15}$. (d) $\frac{0}{a}$, a any counting number.

For (c) and (d), 0 would also be correct.

17. Yes, each method relies on the subtraction of whole numbers.

18. (a) $\frac{21}{20}$ (b) $\frac{860}{387}$ (c) $\frac{12}{6}$ or $\frac{2}{1}$ (d) $\frac{1}{1}$ (e) $\frac{9}{4}$

19. (b), (c), (e), and (f).

20. (a) $^-4$ (b) $^-38$ (c) 38 (d) 4

21. (a) 91 (b) $^-91$ (c) $^-91$ (d) 91

22. (a) 27 (b) $^-27$ (c) $^-27$ (d) 27

Chapter 7

PRIMES AND FACTORS

Introduction

Chapter 5 indicated the need for new numbers to answer certain questions that counting numbers cannot answer, and introduced the rational numbers. The counting numbers were identified with certain rational numbers. Chapter 6 defined binary operations on the rational numbers.

In this chapter we shall take another look at whole numbers, investigating a collection of ideas not only interesting in themselves, but useful in the study of number systems.

7.1 Whole Numbers - A New Look

We can ask questions in terms of whole numbers that cannot be answered with whole numbers. For example, the equation $5x = 9$ stated with whole numbers cannot be solved with a whole number. This situation led to the development of the positive rational numbers. Now, we shall back up a bit and examine the whole numbers in some detail. That some equations of the form $bx = a$, a and b whole numbers, have solutions among the whole numbers whereas others do not, is in itself intriguing.

Note the following equations.

Equations (Stated with whole numbers)	Solution Set (Restricted to whole numbers)
$3x = 3$	$\{1\}$
$2x = 6$	$\{3\}$
$5x = 420$	$\{84\}$
$3x = 7$	$\emptyset$
$5x = 9$	$\emptyset$

That $2x = 6$ has a whole number solution, 3, but that $5x = 9$ has no whole number solution suggests a study of multiplicative properties of whole numbers. Can we distinguish those pairs of whole numbers a, b for which solutions of $bx = a$ can be found?

Let us examine how whole numbers can be expressed as products of other whole numbers.

> Given the numbers a and b, we say that b is a factor of a if and only if a whole number c can be found such that
>
> $$bc = a .$$

For example, if $a = 10$ and $b = 5$, then we have

$$5c = 10 .$$

We find that c equals the whole number 2.

$$5 \cdot 2 = 10 .$$

Hence we conclude that 5 is a factor of 10. By use of the commutative property, we can rewrite the last equation as $2 \cdot 5 = 10$ indicating that 2 is also a factor of 10.

The concept of factor for numbers is only interesting if a restriction is made in the definition. Let us see what would happen if the adjective "whole" were omitted from the definition. If this were done, then any non-zero number would be a factor of every number. For example:

17 would be a factor of 100 since $17 \cdot \frac{100}{17} = 100$;

12 would be a factor of 18 since $12 \cdot \frac{3}{2} = 18$;

$\frac{9}{10}$ would be a factor of $\frac{2}{3}$ since $\frac{9}{10} \cdot \frac{20}{27} = \frac{2}{3}$.

Thus in the concept of factoring, there is always a restriction implied. Here our restriction is to whole numbers.

Because the same idea arose in different branches of mathematics, other language besides "factor" is also used: for instance: "divides", "divisor", and "multiple of". "Divides" means that division produces a quotient without a remainder. Thus, from $5 \cdot 2 = 10$ we say that 5 "divides" 10; that 5 is a "divisor" of 10; and that 10 is a "multiple of" 5.

Each whole number has many names. For example, the number 24 may be written as the product of two whole numbers. When 24 is written as the product of two whole numbers, the equality is called a product expression.

All product expressions of 24 are:

$1 \times 24 = 24$

$2 \times 12 = 24$

$3 \times 8 = 24$

$4 \times 6 = 24$

From these product expressions, we name the possible factors of 24 as 1, 2, 3, 4, 6, 8, 12, 24. The factors of 24 determine the whole number replacements for b in the equation $bx = 24$ that give whole number solutions.

What equations of the form $bx = a$ can we make using $a = 24$ such that b and x are whole numbers?

$1x = 24$ $\qquad$ $6x = 24$

$2x = 24$ $\qquad$ $8x = 24$

$3x = 24$ $\qquad$ $12x = 24$

$4x = 24$ $\qquad$ $24x = 24$

Does this mean that these are the only questions of the form $bx = a$, $a = 24$, that we can answer with x a whole number? Yes, because all factors of 24 were used as replacements for b.

Suppose other whole numbers are used as replacements for b in $bx = 24$ as shown below.

$5x = 24$ $\qquad$ $10x = 24$

$7x = 24$ $\qquad$ $30x = 24$

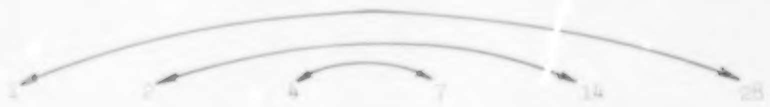
While we know that each of these equations has a solution which is a rational number, none has a solution among the whole numbers. Thus in $bx = 24$, b and x whole numbers, b may have replacements 1, 2, 3, 4, 6, 8, 12, 24, but may not have other replacements such as 5, 7, 10, 30.

In general, we see that if a and b are whole numbers and we want x to be a whole number in $bx = a$, then b must be a factor of a.

Class Exercises

For exercises 1-3, a, b, and x are restricted to counting numbers with a a multiple of b.

1. For $a = 28$, list the factors of a and write all equations of the form $bx = a$, for which x has a solution that is a whole number.
2. Factors of a number can be paired so that their product is the given number. For example, the factors of 28 may be paired.

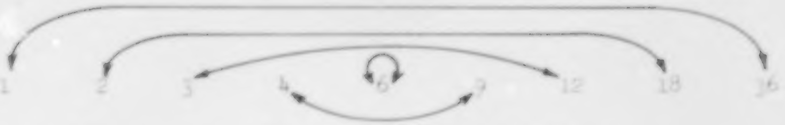


Another arrangement is seen in the factor pairs of 36. Product expressions for 36 are:

$$1 \times 36 = 36$$
$$2 \times 18 = 36$$
$$3 \times 12 = 36$$
$$4 \times 9 = 36$$
$$6 \times 6 = 36$$

The factors for 36 are: 1, 2, 3, 4, 6, 9, 12, 18, 36.

The factor pairs are:



List the factors and indicate the factor pairs for:

a. 18   b. 32   c. 25

3. For each part of Exercise 2, how many equations of the form $bx = a$ can be written so that $\underline{x}$ is a counting number and $\underline{a}$ has the value indicated?

Let us explore the role that zero plays when factors and products are under consideration.

Since the product of zero and any number is zero, we can rule out equations such as

$$0 \cdot x = 17 .$$

No whole number $\underline{x}$ makes this statement true. Hence, 0 is not a factor of 17. Consider the equation

$$0 \cdot x = 0 .$$

Every whole number x makes the statement true. Hence, 0 is a factor of 0. Last, consider the equation

$$17 \cdot x = 0 .$$

The whole number 0 makes this sentence true. Hence, 17 is a factor of 0. However, this is not a very exciting fact since we must therefore conclude that every number is a factor of 0.

Since 17 is a factor of 0, we say that 17 divides 0. Since 0 is not a factor of 17, we say that 0 does not divide 17. However, while we agree to say that 0 is a factor of 0, we do not, in this case, say that 0 divides 0.

One is a factor of any number. In fact, certain counting numbers can be expressed as a product only of themselves and 1. That 1 is a factor is so obvious that it is frequently omitted in listing factors of a number. In some cases, however, this is the only way that a whole number can be expressed as a product. For example:

$$7 = 7 \times 1 \qquad 3 = 3 \times 1 \qquad 13 = 13 \times 1$$

In summary, regarding zero and one as factors, we say:

> Zero is not a factor of any whole number except itself.
> One is a factor of every whole number.

7.2 Prime Numbers

In the preceding section, we studied factors. In this section we introduce several classifications of the counting numbers and learn how such classifications may help in calculating with rational numbers.

One such classification consists of even numbers and odd numbers. A number is even if it can be expressed in the form $2n$, n a whole number. Zero is an even number in that zero may be expressed in the form $2n$: $0 = 2(0)$. Also base ten numerals ending with the digit 0 represent even numbers, since each may be expressed in the form $2n$. For example,

$$30 = 2 \cdot 15$$
$$3000 = 2 \cdot 1500$$

A number is odd if it can be expressed in the form $2n + 1$. For example,

$$1 = 2 \cdot 0 + 1 \qquad 9 = 2 \cdot 4 + 1$$
$$3 = 2 \cdot 1 + 1 \qquad 11 = 2 \cdot 5 + 1$$
$$5 = 2 \cdot 2 + 1 \qquad 79 = 2 \cdot 39 + 1$$
$$7 = 2 \cdot 3 + 1$$

Some textbooks state that if a whole number is divisible by 2, then it is an even number; and if a whole number is not divisible by 2, then it is an odd number. This seemingly rudimentary classification of the whole numbers into these two classes has many uses (remember the unicursal problems?), and will be used later to prove $\sqrt{2}$ is not a rational number.

Whole numbers may thus be classified into two sets:

$$O = \{1, 3, 5, 7, \ldots\}$$
$$E = \{0, 2, 4, 6, \ldots\}$$

For the next classification we shall consider only the counting numbers. If we examine each of the first fourteen counting numbers, we find several definite patterns among the sets of factors. We list all factors, as shown.

Counting Number	Factors
1	1
2	1, 2
3	1, 3
4	1, 2, 4
5	1, 5
6	1, 2, 3, 6
7	1, 7
8	1, 2, 4, 8
9	1, 3, 9
10	1, 2, 5, 10
11	1, 11
12	1, 2, 3, 4, 6, 12
13	1, 13
14	1, 2, 7, 14

Some numbers, like 2, 3, 5, 7, and 11 can be expressed as a product of only themselves and 1. These numbers are called *primes*. Other numbers such as 4, 6, 9, and 14 have factors different from themselves and 1. Such numbers are *composite* numbers. (For convenience in stating theorems, the number 1 is considered neither a prime number nor a composite number.) This discussion leads to the definitions in SMSG *Mathematics for Junior High School*, Volume I, which are repeated here.

A *prime number* is a counting number, other than 1, which is divisible only by itself and 1.

A *composite number* is a counting number which is divisible by a smaller counting number different from 1. Thus a composite number is a counting number different from 1 which is not a prime.

When we speak of the **complete factorization** of a number, we refer to the number written as a product of prime factors. Frequently it is expedient to think of a composite number as a product of its factors. If 175 is written as $5 \times 5 \times 7$, it is shown as the product of its prime factors; this is the complete factorization of 175. Regardless of whether we divide first by 5 or by 7, we complete the factorization with the same prime factors: the only difference is order.

$$175 = 25 \times 7 = 5 \times 5 \times 7$$
$$175 = 5 \times 35 = 5 \times 7 \times 5$$
$$175 = 35 \times 5 = 7 \times 5 \times 5$$

These indicated products are all equal. Recall that changing the order of the factors in multiplication can be accomplished by using the associative and commutative properties.

The property we have just observed, that the complete factorization of a number is unique, is called the Unique Factorization Property.

> Unique Factorization Property: Every counting number greater than 1 can be written as a product of primes. Except for order, this factorization is unique.

This property is sometimes called the Fundamental Theorem of Arithmetic. Examples readily convince students of the validity of this theorem. (Various proofs exist and may be found in any book on number theory.)

It is frequently convenient to use the exponential form in the complete factorization of a number. For example,

$$144 = 3 \times 3 \times 2 \times 2 \times 2 \times 2 = 3^2 \times 2^4 .$$

In review, we note that the set of counting numbers may be partitioned into three subsets:

The set of prime numbers
The set of composite numbers
The set containing the number 1

Class Exercises

4. Find the smallest prime factor of
   (a) 135 (b) 539 (c) 484

5. Give the complete factorization of
   (a) 26 (b) 210 (c) 47

6. Give the complete factorization of 600 in exponential form.

7.3 Least Common Multiple, Greatest Common Factor

The study of composite numbers and their factorizations leads us to least common multiples and greatest common factors

The least common multiple (l.c.m.) of a set of counting numbers is the smallest counting number which is a multiple of each number of the set. Consider the numbers 6 and 16 and the set of multiples for each.

Set of multiples of 6: {6, 12, 18, 24, 30, 36, 42, 48, 54, 60, ...}

Set of multiples of 16: {16, 32, 48, 64, 80, 96, ...}

By inspecting the two sets of multiples we see that 48 is the smallest multiple common to both. Hence the l.c.m. of 6 and 16 is 48.

The set of all common multiples of 6 and 16 is

{ 48, 96, 144, 192, ... }.

Note that a set of common multiples for two numbers is always an infinite set: there is no greatest common multiple.

The greatest common factor (g.c.f.) of a set of counting numbers is the largest counting number that is a factor of each member of the set.

Again consider the numbers 6 and 16 but this time with the set of factors for each.

Set of factors of 6: {1, 2, 3, 6}
Set of factors of 16: {1, 2, 4, 8, 16}

By inspecting the two sets of factors we see that 2 is the largest factor common to both. Hence the g.c.f of 6 and 16 is 2. The set of all common factors of 6 and 16 is

{ 1, 2 } .

The set of common factors for two numbers is always finite: there is always a greatest common factor.

Another method for finding the l.c.m. and the g.c.f of two numbers utilizes their complete factorizations. The complete prime factorizations of 36 and 120 are given here.

$$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$$

$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^3 \cdot 3 \cdot 5$$

The l.c.m. must contain all the different prime factors of each number and these factors must occur as frequently as the greater number of times they occur in either of the factorizations. Thus the least common multiple of 36 and 120 is

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^3 \cdot 3^2 \cdot 5 = 360 .$$

The g.c.f. must contain only those prime factors common to each number and these factors must occur only as frequently as the lesser number of times that they occur in the factorizations. Thus the greatest common factor of 36 and 120 is $2 \cdot 2 \cdot 3 = 2^2 \cdot 3^1 = 12$.

Class Exercises

7. Find the l.c.m. for each pair of numbers.
   (a) 8 and 12 (b) 14 and 35

8. Find the g.c.f. for each pair of numbers.
   (a) 48 and 80 (b) 16 and 36

9. Give the complete factorization of 24 and 90 in exponential form. Then write their l.c.m. and g.c.f. in exponential form.

10. What is the greatest common factor of any two prime numbers p and q ? What is the least common multiple of the two primes?

Let us factor completely the two numbers 32 and 20.

$$32 = 2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

$$20 = 2 \times 2 \times 5 = 2^2 \times 5$$

From their complete factorizations, we find:

the l.c.m. of 32 and 20 is $2^5 \times 5 = 160$;

the g.c.f. of 32 and 20 is $2^2 = 4$.

To take this a bit further, the product of the l.c.m. and the g.c.f. is

$$160 \times 4 = 640 .$$

Compare this with the product of the original numbers 32 and 20.

$$32 \times 20 = 640$$

For two counting numbers, it is always true that their product is the same as the product of their l.c.m. and g.c.f.

If $\underline{m}$ and $\underline{n}$ are any two counting numbers, the product of their l.c.m. and g.c.f. is $m \times n$.

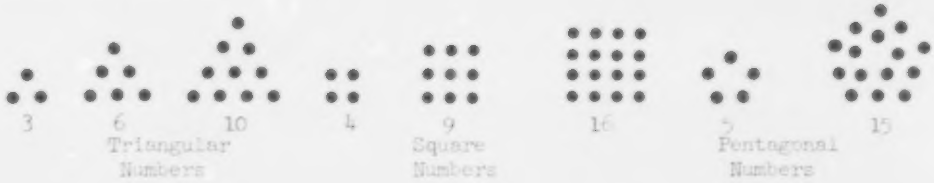
Class Exercises

11. Find the least common multiple and the greatest common factor of 64 and 36. Compare the product of their l.c.m. and g.c.f. with the product of the original numbers.

12. Two bells are set so that their time interval for striking is different.
    (a) One bell strikes every 3 minutes and the second strikes every 5 minutes. If both bells strike together at 12:00 noon, when will they strike together again?
    (b) One bell strikes every 6 minutes and the second bell every 15 minutes. If they both strike at 12:00 noon, when will they strike together again?
    (c) Find the l.c.m. of 3 and 5, and the l.c.m. of 6 and 15. Compare these with answers of parts (a) and (b).

7.4 Some Historical Comments

The ancient Greeks loved to study numbers. They gave fanciful and mystical names and interpretations to numbers with certain special properties. They spoke of triangular, square, and pentagonal numbers because of their geometric properties.



They also spoke of perfect and amicable numbers. These numbers had special properties determined by their factors. Let us look at one set of these mystical numbers in more detail.

Consider the table of factors of some of the whole numbers as shown below. We immediately recognize those numbers with only two factors as being prime.

n	Factors of n	n	Factors of n
1	1	11	1, 11
2	1, 2	12	1, 2, 4, 6, 12
3	1, 3	13	1, 13
4	1, 2, 4	14	1, 2, 7, 14
5	1, 5	15	1, 3, 5, 15
6	1, 2, 3, 6	16	1, 2, 4, 8, 16
7	1, 7	17	1, 17
8	1, 2, 4, 8	18	1, 2, 3, 6, 9, 18
9	1, 3, 9	19	1, 19
10	1, 2, 5, 10	20	1, 2, 4, 5, 10, 20

If, for each whole number, we take the sum of all its factors except the number itself, we find that the sums fall into three groups. Certain sums are greater than their respective numbers; other sums are smaller than their corresponding numbers. But in a few cases, the sum is the same as the number. Such a number is called a perfect number.

Six is a perfect number, since

$$6 = 1 + 2 + 3 .$$

Another perfect number is 496, since

$$496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 .$$

Only a few perfect numbers have been found.

In the table above showing the factors of n, do you notice that some numbers have exactly two distinct factors whereas other numbers have more than two? Observe that 1 is a factor of every counting number. Do you notice patterns for the occurence of 2 as a factor? of 3 as a factor? of 5 as a factor?

Amicable numbers are pairs of numbers with the following property: For each number the sum of all its factors except the number itself, equals the other number. The numbers 220 and 284 are examples of amicable numbers.

The factors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110, 220.

$$1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = \underline{284} .$$

The factors of 284 are 1, 2, 4, 71, 142, 284.

$$1 + 2 + 4 + 71 + 142 = \underline{220} .$$

These are the smallest amicable numbers. Another pair, found by Fermat, is 17,296 and 18,416.

Class Exercises

13. Show that 28 is a perfect number.

14. Can you find another perfect number p such that $8000 < p < 8130$?

15. When the sum of all factors of a number except the number itself is not large enough to make a perfect number, the sum is said to be "deficient". Sums too large to make a perfect number are said to be "abundant". Indicate which of the following numbers have deficient sums and which have abundant sums.

(a) 10 (b) 16 (c) 20 (d) 36

The properties of prime numbers have challenged mathematicians throughout the ages. Euclid, the famous Greek who wrote the first geometry textbooks called the Elements about 300 B.C., was able to prove that there are an infinite number of primes.

Eratosthenes, who lived about 225 B.C., and is famous for his indirect measurement of the diameter of the earth, also studied primes. He developed a method called the "sieve of Eratosthenes" for finding primes by sifting out composite numbers. The method uses the fact that every second counting number from 2 is composite and has a factor 2; every third counting number from 3 is composite and has a factor 3; every fifth counting number after 5 has a factor 5, and so forth. To find the primes less than or equal to 100 by this method, first list in order the counting numbers from 1 to 100. Cross out every second number after the prime 2 since these are all composite numbers that contain the prime factor 2. Next cross out every third number after the prime 3 since they all contain the prime factor 3. The number 4 has already been crossed out and is therefore not prime. The next number not crossed out is the prime 5. Every fifth number after the prime 5 is then crossed out. This eliminates all composites that are multiples of 5. In like manner 6, 8, 9, and 10 have already been eliminated as composites, while 7 and 11 are found to be prime. The table should now look like the following. From our work with factor pairs, we know that every composite number 100 or less with a factor greater than 10 has a corresponding factor less than 10. All composite numbers with factors less than 10 have already been eliminated.

Thus, all composite numbers in the table have been crossed out: only the primes equal to or less than 100 and the number 1 remain.

Sieve of Eratosthenes for the numbers from 1 through 100:

•1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

•Remember! 1 is not a prime number.

While this method is useful in locating primes equal to or less than 100, it cannot be used to locate all primes. Indeed, some 2000 years after Eratosthenes, mathematicians have still not found a method for finding all primes.

Number Theory is possibly the oldest branch of higher mathematics. Part of its fascination over the years has been the ease with which problems may be stated. Many problems need only some knowledge of arithmetic and of primes to be stated. The solution of some of these problems may be found easily at this level; others require greater ingenuity. Some, though simply stated, may require mathematical reasoning and techniques of the highest order.

A very elementary theorem to prove is the one already given:

> For whole numbers m and n the product of their
> least common multiple and their greatest common factor
> is equal to the product of m and n.

At the other extreme is the problem known as the Goldbach Conjecture:

> Every even number greater than 4 may be written as
> the sum of two primes.

Though some progress has been made on this problem, it has resisted a complete answer for over 150 years. It is still only a conjecture.

An even older problem dating back at least to Euclid concerns the perfect numbers. All the known perfect numbers are even numbers. No one has ever succeeded in finding an odd perfect number nor has anyone been able to show that there are no odd perfect numbers. It is clear that no prime is a perfect number. While it is not quite so apparent, the product of two odd primes cannot be a perfect number. Still harder to prove, but true, is that the product of three odd primes is not a perfect number. This is the sort of information that has been collected on this problem, but is still a long way from a solution.

Class Exercise

Many mathematicians have tried to prove that:

> There are no integers x, y, and z for which $x^n + y^n = z^n$ if $n > 2$. (Known as Fermat's Last Theorem, it has not been proved.)

16. Using your knowledge of the Pythagorean property, experiment with $x^2 + y^2 = z^2$ by finding replacements for x, y, and z. Try some of these number triples in $x^n + y^n = z^n$ with $n = 3$ or $n = 4$. Are you successful in finding some that work?

7.9 Positive Rational Numbers - Role of Factors

We pause long enough to see how the factoring of composite numbers may be used to improve the mechanics of operating with numbers that are expressed in fractional form. The mechanical aspects of addition leave room for variations. Our motivation for addition of rational numbers essentially depended upon finding a common denominator which is a common multiple of the denominators. The common denominator used to add $\frac{a}{b}$ and $\frac{c}{d}$ was bd; in some cases this is the smallest common denominator that may be used.

Let us consider several examples.

Example 1. To find $\frac{3}{5} + \frac{2}{3}$ we note that both denominators are prime numbers. The common multiple, 15, is also the least common multiple. Thus:

$$\begin{aligned}\frac{3}{5} + \frac{2}{3} &= \frac{3}{5} \cdot \frac{3}{3} + \frac{2}{3} \cdot \frac{5}{5} \\ &= \frac{9}{15} + \frac{10}{15} \\ &= \frac{19}{15}\ .\end{aligned}$$

Example 2. To add $\frac{5}{24}$ and $\frac{5}{18}$ we may use as a common denominator $24 \times 18 = 432$. The number 432 is a common multiple of 24 and 18. Following the method of Example 1, we have

$$\begin{aligned}\frac{5}{24} + \frac{5}{18} &= \frac{5}{24} \cdot \frac{18}{18} + \frac{5}{18} \cdot \frac{24}{24} \\ &= \frac{90}{432} + \frac{120}{432} \\ &= \frac{210}{432} .\end{aligned}$$

A quick observation indicates that since the numerator and denominator end in 0 and 2, the fraction can be reduced.

Rather than using this relatively large number, 432, as a common multiple of the denominators of $\frac{5}{24}$ and $\frac{5}{18}$, we can simplify our computation using the least common multiple of 24 and 18. Because both denominators are composite numbers, we factor the numbers to determine their least common multiple.

$$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$$

$$18 = 2 \cdot 3 \cdot 3 = 2 \cdot 3^2$$

The l.c.m. of 24 and 18 is $2^3 \cdot 3^2 = 8 \cdot 9 = 72$.

$$\begin{aligned}\frac{5}{24} + \frac{5}{18} &= \frac{5}{24} \cdot \frac{3}{3} + \frac{5}{18} \cdot \frac{4}{4} \\ &= \frac{15}{72} + \frac{20}{72} \\ &= \frac{35}{72}\end{aligned}$$

Following the equation method given in Chapter 6, we may add the numbers $\frac{5}{24}$ and $\frac{5}{18}$ as shown below.

Let $x = \frac{5}{24}$ and $y = \frac{5}{18}$

Then $24x = 5$ and $18y = 5$.

To suggest the distributive law we multiply the first equation by 18 and the second one by 24 as shown at the left. We may just as well multiply the first equation by 3 and the second by 4 as shown at the right. In either case, the value for $x + y$ can be found. The second method, utilizing the least common multiple, simply gives the result in a more simplified form.

$$
\begin{array}{rcl}
435x & = & 90 \\
435y & = & 120 \\
\hline
435x + 435y & = & 90 + 120 \\
435\,(x + y) & = & 210 \\
x + y & = & \frac{210}{435}
\end{array}
\qquad\qquad
\begin{array}{rcl}
72x & = & 15 \\
72y & = & 20 \\
\hline
72x + 72y & = & 15 + 20 \\
72\,(x + y) & = & 35 \\
x + y & = & \frac{35}{72}
\end{array}
$$

This last method is not suggested for the seventh grader. It is given here to emphasize again how rational numbers can be treated through equations. However, the similarity of this method with the first should be apparent.

Example 3. Last, we find the sum of three addends: $\frac{7}{24}$, $\frac{11}{36}$, and $\frac{5}{72}$.

$$
\begin{aligned}
24 &= 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3 \\
36 &= 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \\
72 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2
\end{aligned}
$$

The l.c.m. of 24, 36, 72 is $2^3 \cdot 3^2 = 72$.

$$
\begin{aligned}
\frac{7}{24} + \frac{11}{36} + \frac{5}{72} &= \frac{7}{24} \cdot \frac{3}{3} + \frac{11}{36} \cdot \frac{2}{2} + \frac{5}{72} \\
&= \frac{21}{72} + \frac{22}{72} + \frac{5}{72} \\
&= \frac{48}{72}
\end{aligned}
$$

The reader, looking at the sum $\frac{48}{72}$, feels immediately that we can reduce the fraction. A quick check of divisibility reveals that both 48 and 72 are divisible by 2 and by 3. However, let us use another procedure. We determine the greatest common factor of 48 and 72.

$$
\begin{aligned}
48 &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^4 \cdot 3 \\
72 &= 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2
\end{aligned}
$$

The g.c.f. of 48, 72 is $2^3 \cdot 3 = 24$.

$$\frac{48}{72} = \frac{24}{24} \cdot \frac{2}{3} = \frac{2}{3}$$

This is the second time in the same example that we have had the opportunity to rename a rational number.

It follows that

$$\frac{7}{24} + \frac{11}{36} + \frac{5}{72} = \frac{2}{3}.$$

It is not always possible to reduce the results as was done in Example 3. Looking at another sum we computed, $\frac{35}{72}$ and $\frac{19}{15}$, we proceed in the same manner.

$$35 = 5 \cdot 7 \qquad\qquad 72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$$

A greatest common factor for 35 and 72 is 1. Hence, the fraction $\frac{35}{72}$ cannot be reduced.

Next, we examine 19 and 15 in the same way.

$$19 = 19 \cdot 1 \qquad\qquad 15 = 3 \cdot 5$$

The greatest common factor for 19 am 15 is 1, since there are no common primes in the complete factorization of these two numbers. Hence, the fraction $\frac{19}{15}$ cannot be reduced.

Two numbers containing no common prime factors are said to be relatively prime to each other. If two relatively prime numbers serve as the numerator and denominator of a fraction, then the fraction is said to be in "lowest terms" and cannot be reduced.

Prime numbers, complete factorization, least common multiples, and greatest common factors apply to the study of counting numbers. The positive rational numbers can be defined in terms of the counting numbers. Hence, it is not surprising to find that we make use of the idea of the least common multiple of counting numbers in finding the "lowest common denominator" when adding fractions. Likewise, we make use of the idea of the greatest common factor of counting numbers in "reducing fractions to lowest terms". The purpose of this section is to illustrate the role of factors and multiples in operating with the positive rational numbers.

Factors and multiples play a role in all four of the fundamental operations with the rational numbers. The use of least common multiples is evident in subtraction with rational numbers, just as in addition. For the operation subtraction with rational numbers, we cite the example $\frac{4}{50} - \frac{4}{75}$.

$$50 = 2 \cdot 5 \cdot 5 = 2 \cdot 5^2$$

$$75 = 3 \cdot 5 \cdot 5 = 3 \cdot 5^2$$

The l.c.m. of 50, 75 is $2 \cdot 3 \cdot 5^2 = 150$.

$$\frac{4}{50} - \frac{4}{75} = \frac{4}{50} \cdot \frac{3}{3} - \frac{4}{75} \cdot \frac{2}{2}$$

$$= \frac{12}{150} - \frac{8}{150} = \frac{\text{[illegible]}}{150}$$

Next, we examine 4 and 150 for their g.c.f.

$$4 = 2 \cdot 2 = 2^2$$
$$150 = 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2$$

The g.c.f. of 4, 150 is 2 .

Hence, $\frac{4}{150} = \frac{2}{2} \cdot \frac{2}{75} = \frac{2}{75}$

and, $\frac{4}{50} - \frac{4}{75} = \frac{2}{75}$.

The most common way to add rational numbers, when they are named as fractions whose denominators differ, is to rename them with the same denominator. Efficiency in renaming these numbers is achieved by using their least common multiple. However, this will not necessarily yield the answer in simplest form. The key idea in renaming a rational number in simplest fraction form is the use of the greatest common factor of numerator and denominator.

Class Exercise

17. a. Combine, as indicated, using complete factorizations as needed.

$$\frac{2}{3} - \frac{2}{77} + \frac{5}{21}$$

b. Check the result for common factors in numerator and denominator.

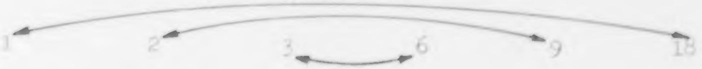
c. Write, in set notation, the set of factors for each denominator in (a). What is the union of these three sets? Compare this answer with the factors determining the l.c.m. used in (a).

Answers to Class Exercises

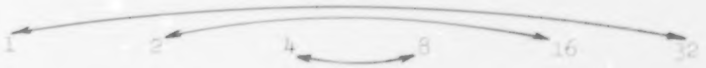
1. 1, 2, 4, 7, 14, 28

$x = 28$ $7x = 28$
$2x = 28$ $14x = 28$
$4x = 28$ $28x = 28$

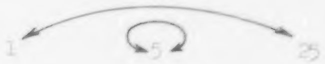
2. a. 1, 2, 3, 6, 9, 18



b. 1, 2, 4, 8, 16, 32



c. 1, 5, 25



3. a. 6   b. 6   c. 3

4. a. 3   b. 7   c. 2

5. a. $26 = 2 \times 13$
b. $210 = 2 \times 3 \times 5 \times 7$
c. $47 = 47 \times 1$ (prime)

6. $600 = 2^3 \times 3 \times 5^2$

7. a. 24   b. 70

8. a. 16   b. 4

9. $24 = 2^3 \times 3$, $90 = 2 \times 3^2 \times 5$
l.c.m. = $2^3 \times 3^2 \times 5$
g.c.f. = 2×3

10. 1 ; pq

11. l.c.m. = 576 g.c.f. = 4

$$576 \times 4 = 2304$$
$$64 \times 36 = 2304$$

12. a. 3, 6, 9, 12, 15, ... --- first bell
5, 10, 15, ... --- second bell
They strike together again in 15 minutes; that is, at 12:15 o'clock.

b. 6, 12, 18, 24, 30, ... --- first bell
15, 30, ... --- second bell
They strike together again in 30 minutes; that is, at 12:30 o'clock.

c. l.c.m. of 3, 5 = 15
l.c.m. of 6, 15 = 30

13. $28 = 1 + 2 + 4 + 7 + 14$

14. $8128 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508$
$1016 + 2032 + 4064$

15. deficient: 10, 16
abundant: 20, 36

16. Some examples are:

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$10^2 + 24^2 = 26^2$$

$x^n + y^n = z^n$, $n > 2$, has intrigued mathematicians for many years. No number triples have been found which make $x^n + y^n = z^n$ true for $n > 2$.

17. a. $3 = 3$ the l.c.m. of 3, 77, 21 is $3 \cdot 7 \cdot 11 = 231$

$77 = 7 \cdot 11$

$21 = 3 \cdot 7$

$$\frac{2}{3} - \frac{2}{77} + \frac{5}{21} = \frac{22 \cdot 7}{33 \cdot 7} - \frac{2 \cdot 3}{77 \cdot 3} + \frac{5 \cdot 11}{21 \cdot 11}$$
$$= \frac{154}{231} - \frac{6}{231} + \frac{55}{231} = \frac{203}{231}$$

b. $203 = 29 \cdot 7$

$231 = 7 \cdot 11 \cdot 3$

g.c.f. of 203, 231 = 7

Hence, $\frac{203}{231} = \frac{29 \cdot 7}{33 \cdot 7} = \frac{29}{33}$.

c. $A = \{3\}$; $B = \{7, 11\}$; $C = \{3, 7\}$.

$(A \cup B) \cup C = \{3, 7, 11\}$

l.c.m. of 3, 77, 21 is $3 \cdot 7 \cdot 11 = 231$

The union of the 3 sets is the same as the set of factors used to determine the l.c.m.

Chapter Exercises

1. Find a complete factorization of each of the following:

   (a) 39 (c) 81 (e) 180 (g) 576

   (b) 60 (d) 98 (f) 258 (h) 2324

2. Find the least common multiple (l.c.m.) and the greatest common factor (g.c.f.) for each pair of numbers.

   (a) 6, 78 (b) 14, 105 (c) 37, 41

3. Copy the following table for counting number N and complete it through N = 30.

N	Factors of N	Number of Factors	Sum of Factors
1	1	1	1
2	1,2	2	3
3	1,3	2	4
4	1,2,4	3	7
5	1,5	2	6
6	1,2,3,6	4	12
7	1,7	2	8
8	1,2,4,8	4	15

   a. Which numbers represented by N in the table above have exactly two factors?

   b. Which numbers N have exactly three factors?

   c. If $N = p^2$ (where p is a prime number), how many factors does N have?

   d. If $N = pq$ (where p and q are different prime numbers), how many factors does N have? What is the sum of its factors?

   e. If $N = 2^k$ (where k is a counting number), how many factors does N have?

4. a. Is it possible to have exactly four composite numbers between two consecutive primes? If so, give an example.

   b. Is it possible to have exactly five consecutive composite numbers between two consecutive primes? If so, give an example.

5. Given the numbers 135, 222, 783, 1065. Without dividing answer the following questions. Then check your answers by dividing.

   a. Which numbers are divisible by 3?
   b. Which numbers are divisible by 6?
   c. Which numbers are divisible by 9?
   d. Which numbers are divisible by 5?
   e. Which numbers are divisible by 15?
   f. Which numbers are divisible by 4?

6. 112 tulip bulbs are to be planted in parallel rows in a garden. Describe all possible arrangements of the bulbs if they are to be planted in straight rows with an equal number of bulbs per row.

7. Ten tulip bulbs are to planted so that there will be exactly five rows with four bulbs in each row. Draw a diagram of this arrangement.

8. Which of the following numbers are divisible by 2?

   a. 1111_{ten}
   b. 1111_{seven}
   c. 1111_{six}
   d. 1111_{three}

Chapter 8

DECIMALS, RATIOS, AND PERCENTS

Introduction

By the time that a youngster reaches the seventh grade, he should have been exposed to the fundamental operations with decimals. Quite often he will be familiar with the algorithms but not with their rationale. Thus, he may know how to "shift" decimal points in division, but still have no idea why he is doing so.

Consequently, the first objective in presenting a unit on decimals in grade seven is to review the fundamental operations in terms of their basic meanings and rationale. This proves to be a non-trivial task, inasmuch as seventh graders all too often feel that they know everything they should or wish to know about decimals, at least insofar as the mechanics are involved. They do not look with favor upon what they consider to be a review of elementary mathematics. It will, therefore, take "salesmanship" to convince them of the importance of understanding what they are doing.

A second major objective for teaching a unit on decimals is that the development of the set of real numbers, together with its properties, is best accomplished through a discussion of decimals. In Chapters 5-7 we have developed the number system through the set of rationals. In this chapter and the next we shall use decimals to explore some of the properties of the set of real numbers.

There are numerous social applications of decimals and percents that can be introduced by the teacher, although most texts now tend to place less emphasis on such applications than has been the case in the past.

We should note here that various textbooks differ on the language that is used to discuss decimals. For example, although some texts refer to "decimal fractions," this terminology will be avoided here.

8.1 Decimal Notation

The notation commonly used for decimals is merely a matter of convenience. Actually, we could have managed if decimals had never been invented, but it would have been far more difficult to compute than is now the case. We also need decimals to help satisfy the demands of the real world. Thus, the experimental scientist does not work with numbers like π and $\sqrt{2}$, but rather with such rational approximations of these numbers as 3.1416 and 1.4142 .

The history of the development of fractions is an interesting one. The ancient Egyptians, for example, wrote all of their fractions as the sum of unit fractions; that is, as fractions with 1 as numerator. (The only exception was the fraction $\frac{2}{3}$ for which a special symbol was used.) For example, they wrote:

two-sevenths as $\frac{1}{4} + \frac{1}{28}$ instead of $\frac{2}{7}$

five-sixths as $\frac{1}{2} + \frac{1}{3}$ instead of $\frac{5}{6}$.

The Rhind Papyrus has a set of tables that show how to express many fractions in terms of unit fractions.

In a sense, our study of decimals begins through an extension of this system. That is, we now wish to represent every fraction in terms of a special set of fractions, namely those with denominators that are powers of ten.

If we consider the set of unit fractions with denominators that are powers of ten, we can see that decimal notation is merely another way of naming the numbers represented by these fractions. For example, we write:

$\frac{1}{10} = .1$ (one-tenth)

$\frac{1}{100} = .01$ (one-hundredth)

$\frac{1}{1000} = .001$ (one-thousandth) .

It is important for youngsters to see that $\frac{1}{10}$ and .1 are two different names for the same number; only their form is different. Again note that we could have gone along very well using the fractional forms; the decimal notation is merely a convenience and not a necessity.

It is well to provide seventh graders with an opportunity to write decimals in expanded form, making use of powers of ten and exponents. They have already done so for whole numbers and now should do likewise for decimals. For example:

$$372.4 = (3 \times 10^2) + (7 \times 10^1) + (2 \times 1) + (4 \times \frac{1}{10})$$

$$\begin{aligned} 3.146 &= (3 \times 1) + (1 \times \frac{1}{10}) + (4 \times \frac{1}{100}) + (6 \times \frac{1}{1000}) \\ &= (3 \times 1) + (1 \times \frac{1}{10^1}) + (4 \times \frac{1}{10^2}) + (6 \times \frac{1}{10^3}) . \end{aligned}$$

The similarity between this last expansion and the corresponding decimal interpretation shown below should be apparent.

$$\begin{array}{rlll} 3. & = 3 \times 1 & = 3 \times 1 \\ .1 & = 1 \times .1 & = 1 \times \frac{1}{10} \\ .04 & = 4 \times .01 & = 4 \times \frac{1}{100} \\ \underline{\quad .006} & = 6 \times .001 & = 6 \times \frac{1}{1000} \\ 3.146 & & \end{array}$$

Recall that the value of the one's place can be written as a power of ten using zero as the exponent.

$$10^0 = 1$$

Thus, if desired, we may write

$$5.67 = (5 \times 10^0) + (6 \times \frac{1}{10^1}) + (7 \times \frac{1}{10^2}) .$$

We can use expanded notation to help us express decimals in fractional form as follows:

a.
$$\begin{aligned} 0.23 &= (2 \times \frac{1}{10}) + (3 \times \frac{1}{100}) \\ &= \frac{2}{10} + \frac{3}{100} \\ &= \frac{20}{100} + \frac{3}{100} \\ &= \frac{23}{100} \end{aligned}$$

b.
$$\begin{aligned} 0.3146 &= (3 \times \frac{1}{10}) + (1 \times \frac{1}{100}) + (4 \times \frac{1}{1000}) + (6 \times \frac{1}{10000}) \\ &= \frac{3}{10} + \frac{1}{100} + \frac{4}{1000} + \frac{6}{10000} \\ &= \frac{3000}{10000} + \frac{100}{10000} + \frac{40}{10000} + \frac{6}{10000} \\ &= \frac{3146}{10000} . \end{aligned}$$

Again we note that we merely have two different ways of naming the same number. Furthermore, we note that a number written with one decimal place represents "tenths," one with two places represents "hundredths," and so forth.

For better classes this is a good place to introduce the concept of a negative exponent. Consider, for example, the expansion for 0.3146:

$$0.3146 = (3 \times 10^{-1}) + (1 \times 10^{-2}) + (4 \times 10^{-3}) + (6 \times 10^{-4}) .$$

Here we note that

$$10^{-1} = \frac{1}{10}\ ; \quad 10^{-2} = \frac{1}{10^2}\ ; \quad 10^{-3} = \frac{1}{10^3}\ ; \quad 10^{-4} = \frac{1}{10^4}\ .$$

In general, $10^{-n} = \frac{1}{10^n}$, where n is a whole number. (Indeed, it is true for any integer $\underline{n}$.)

With this last definition we have now given a meaning to any integer as an exponent, be it positive, zero, or negative. The familiar rules for computing with exponents can now be extended to include all integers as exponents.

For all integers $\underline{a}$ and $\underline{b}$,

$$n^a \times n^b = n^{a+b}$$

$$n^a \div n^b = n^{a-b}\ .$$

This is also an excellent opportunity to extend a student's understanding of decimal notation by considering other base notations as well. The pattern is the same: however, instead of powers of ten, we use powers of whatever base has been employed. Here are several examples of expanded notation in other bases. In each case the numerals in the expansion are written in base ten notation. However, we should be careful not to call the numerals on the left "decimals" since this would imply base ten.

$$243.234_{\text{five}} = (2 \times 5^2)+(4 \times 5^1)+(3 \times 1)+(2 \times \tfrac{1}{5})+(3 \times \tfrac{1}{5^2})+(4 \times \tfrac{1}{5^3})$$

$$.4632_{\text{seven}} = (4 \times \tfrac{1}{7}) + (6 \times \tfrac{1}{7^2}) + (3 \times \tfrac{1}{7^3}) + (2 \times \tfrac{1}{7^4})$$

We can express these numerals as base ten numerals merely by completing the indicated computation.

$$\begin{aligned} .342_{\text{five}} &= (3 \times \tfrac{1}{5}) + (4 \times \tfrac{1}{5^2}) + (2 \times \tfrac{1}{5^3}) \\ &= \tfrac{3}{5} + \tfrac{4}{25} + \tfrac{2}{125} \\ &= \tfrac{97}{125} \end{aligned}$$

Since $\frac{97}{125} = \frac{776}{1000}$, we may write:

$$.342_{\text{five}} = .776_{\text{ten}}\ .$$

Class Exercises

Write each of the following in expanded notation.

1. 274.58

3. 32.43_{five}

2. 9.0875

4. 2.0416_{seven}

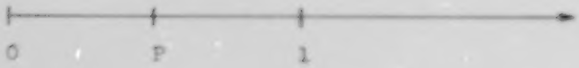
Write as a numeral in base ten notation:

5. $.243_{five}$

6. 32.43_{five}

7. Write each of the following as a power of 2 using negative exponents.
   (a) $\frac{1}{128}$ (b) $\frac{1}{2}$ (c) $\frac{1}{16}$ (d) $\frac{1}{8}$ (e) $\frac{1}{64}$

8. In the following figure the point P indicates the midpoint of the interval from 0 to 1. Name the coordinate of this point with a numeral in:
   (a) base two
   (b) base four
   (c) base eight
   (d) base ten



8.2 Operations with Decimals

Normally one should expect seventh graders to know how to add, subtract, multiply, and divide rational numbers written as decimals. In grade seven we wish to provide opportunities for the maintenance of skills. At the same time it is important that we stress basic meanings and understandings.

In this section we shall review briefly the manner in which the fundamental operations with decimals may be treated in the seventh grade.

Addition

The distributive property is needed here and should be reviewed first. Recall that this property states that for all numbers a, b, and c, we have:

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Now suppose we wish to add two numbers written in decimal form: for example, 0.23 + 0.64 . Certainly the seventh grader will know that one needs to "line up" the decimal points and add, but may not know why we proceed in this manner. We can justify this process by expressing the numbers in fractional form and then using the distributive property.

$$0.23 = \frac{23}{100} = 23 \times \frac{1}{100} \qquad \text{and} \qquad 0.64 = \frac{64}{100} = 64 \times \frac{1}{100}$$

Therefore,

$$\begin{aligned} 0.23 + 0.64 &= (23 \times \frac{1}{100}) + (64 \times \frac{1}{100}) \\ &= (23 + 64) \times \frac{1}{100} \\ &= 87 \times \frac{1}{100} \\ &= \frac{87}{100} \\ &= 0.87 \ . \end{aligned}$$

We then show that this same result can be obtained far more conveniently by writing one numeral below the other.

$$\begin{array}{r} 0.23 \\ \underline{0.64} \\ 0.87 \end{array}$$

In this form we are adding the number in the $\frac{1}{10}$ place in the first addend to the number in the $\frac{1}{10}$ place in the second addend, and so on. That is:

$$\begin{aligned} 0.23 &= (2 \times \frac{1}{10}) + (3 \times \frac{1}{100}) \\ 0.64 &= (6 \times \frac{1}{10}) + (4 \times \frac{1}{100}) \\ 0.23 + 0.64 &= (2 \times \frac{1}{10}) + (6 \times \frac{1}{10}) + (3 \times \frac{1}{100}) + (4 \times \frac{1}{100}) \\ &= (2 + 6) \times \frac{1}{10} + (3 + 4) \times \frac{1}{100} \\ &= (8 \times \frac{1}{10}) + (7 \times \frac{1}{100}) \\ &= \frac{8}{10} + \frac{7}{100} \\ &= \frac{80}{100} + \frac{7}{100} \\ &= \frac{87}{100} \\ &= 0.87 \ . \end{aligned}$$

Note how the distributive property has been used twice. We could also have justified the addition in this manner:

$$\begin{aligned} 0.23 + 0.64 &= 23(.01) + 64(.01) \\ &= .01(23 + 64) \\ &= .01(87) = 0.87 \ . \end{aligned}$$

Note that this latter justification implies that we know that the product $.01 \times 87$ is equal to 0.87. However, this can always be explained by returning to fractional notation:

$$\begin{aligned} .01 \times 87 &= \frac{1}{100} \times 87 \\ &= \frac{87}{100} \\ &= 0.87 \ . \end{aligned}$$

When regrouping ("carrying") is involved, we can justify the usual process as follows:

Procedure:	Justification:
$\begin{array}{r} 0.75 \\ +\ 0.50 \\ \hline 1.25 \end{array}$	$\begin{aligned} 0.75 + 0.50 &= (75 \times \frac{1}{100}) + (50 \times \frac{1}{100}) \\ &= (75 + 50) \times \frac{1}{100} \\ &= (125) \times \frac{1}{100} \\ &= \frac{125}{100} \\ &= \frac{100}{100} + \frac{25}{100} \\ &= 1 + \frac{25}{100} \\ &= 1.25 \end{aligned}$

Subtraction

The subtraction process can be justified in much the same manner as addition and therefore need not be explored in great detail. For example:

$$\begin{aligned} 0.82 - 0.37 &= (82 \times \frac{1}{100}) - (37 \times \frac{1}{100}) \\ &= (82 - 37) \times \frac{1}{100} \\ &= 45 \times \frac{1}{100} \\ &= \frac{45}{100} \\ &= 0.45 \ . \end{aligned}$$

Again, the development makes use of the distributive property.

Using fractions, the rationale of the subtraction process can be illustrated as follows:

$$\begin{array}{r} 0.82 \\ -\ 0.37 \\ \hline \end{array} \quad = \quad \begin{array}{r} \frac{8}{10} + \frac{2}{100} \\ -\frac{3}{10} - \frac{7}{100} \\ \hline \end{array}$$

$$= \quad \begin{array}{l} \frac{7}{10} + \frac{1}{10} + \frac{2}{100} \\ -\frac{3}{10} \qquad - \frac{7}{100} \\ \hline \end{array}$$

$$= \quad \begin{array}{l} \frac{7}{10} + \frac{10}{100} + \frac{2}{100} \\ -\frac{3}{10} \qquad - \quad \frac{7}{100} \\ \hline \end{array}$$

$$= \quad \begin{array}{l} \frac{7}{10} + \frac{12}{100} \\ -\frac{3}{10} - \frac{7}{100} \\ \hline \end{array}$$

$$\frac{4}{10} + \frac{5}{100} = \frac{40}{100} + \frac{5}{100} = \frac{45}{100} = 0.45 \ .$$

Multiplication

The process of multiplication with decimals may also be developed through the use of fractions. Again, it is important to realize that decimal notation is merely a convenience, not a necessity, and that we could get along quite well using only fractions. For example, let us consider the product 0.3×0.25 in fractional form:

$$\begin{aligned} 0.3 \times 0.25 &= (3 \times \frac{1}{10}) \times (25 \times \frac{1}{10^2}) \\ &= (3 \times 25) \times (\frac{1}{10} \times \frac{1}{10^2}) \\ &= 75 \times (\frac{1}{10^3}) \\ &= 0.075 \ . \end{aligned}$$

Notice the use of the associative and commutative property of multiplication in going from the first to the second step.

When we see multiplication of decimals worked out in fractional form we begin to see why we add the number of decimal places in the two factors in order to find the number of decimal places in the product. In the preceding example we multiplied a number expressed in tenths by one expressed in hundredths. In fractional form we found the product:

$$\frac{1}{10^1} \times \frac{1}{10^2} = \frac{1}{10^3} .$$

This suggests the reason for the usual rule of adding the number of decimal places: we are, in reality, adding exponents that are powers of 10. It may be helpful to see this process using negative exponents.

$$\begin{aligned} 0.3 \times 0.25 &= (3 \times 10^{-1}) \times (25 \times 10^{-2}) \\ &= (3 \times 25) \times (10^{-1} \times 10^{-2}) \\ &= 75 \times 10^{-3} \\ &= 0.075 \end{aligned}$$

We need to be careful, however, in our treatment of zeros. Thus, according to the preceding discussion, 0.4×0.75 produces a product expressed in thousandths.

$$\begin{aligned} 0.4 \times 0.75 &= (4 \times \tfrac{1}{10}) \times (75 \times \tfrac{1}{100}) \\ &= (4 \times 75) \times (\tfrac{1}{10} \times \tfrac{1}{10^2}) \\ &= 300 \times (\tfrac{1}{10^3}) \\ &= 0.300 \end{aligned}$$

However, we usually express this product as 0.3: that is:

$$\begin{aligned} 0.300 &= \frac{300}{1000} \\ &= \frac{3}{10} \\ &= 0.3 \;. \end{aligned}$$

Class Exercises

9. Find the sum $0.45 + 0.83$ by using the fractional approach given in this section.

10. Find the difference $0.58 - 0.29$ by using the fractional approach.

11. Find the product 0.23×0.45 by using
    (a) a fractional approach;
    (b) negative exponents.

Division

The approach to division with decimals should also be built upon the assumption that the seventh grader has been exposed to the topic, but needs a fresh look at the rationale of the process as well as practice with the operation. We may begin by assuming that the student knows how to divide with whole numbers. Therefore, if we are able to legitimately convert a division problem that involves decimals to an equivalent one involving whole numbers, then we shall be in good shape. Consider, for example, the quotient $53.75 \div 0.5$. Written in fractional form we have:

$$53.75 \div 0.5 \longrightarrow \frac{53.75}{0.5} \ .$$

Now we multiply this fraction by $\frac{100}{100}$.

$$\begin{aligned} \frac{53.75}{0.5} \times \frac{100}{100} &= \frac{53.75 \times 100}{0.5 \times 100} \\ &= \frac{5375}{50} \end{aligned}$$

Thus, our division problem is reduced to one that involves whole numbers only. We divide and find our quotient to be $107\frac{1}{2}$, or 107.5 :

$$0.5\overline{)53.75} \longrightarrow \begin{array}{r} 107 \\ 50\overline{)5375} \\ \underline{50} \\ 375 \\ \underline{350} \\ 25 \end{array}$$

$$53.75 \div 0.5 = 107\frac{25}{50} = 107.5 \ .$$

From our knowledge of multiplication of decimals, we can check the division by verifying that $0.5 \times 107.5 = 53.75$.

We then proceed to shorten this process somewhat by "shifting" the decimal point so that only the divisor is a whole number:

$$0.5\overline{)53.75} \longrightarrow 5\overline{)537.5} \; .$$

This is legitimate in that we are really multiplying both dividend and divisor by 10. That is:

$$\frac{53.75}{0.5} \times \frac{10}{10} = \frac{537.5}{5} \; .$$

Now all we need to justify is the location of the decimal point in the quotient. We do so by noting that when the divisor is a whole number, then the dividend and the quotient must have the same number of decimal places. (This follows from the fact that the product of the quotient and the divisor gives the dividend.) Since our revised dividend is expressed in tenths, then the quotient must also be in tenths. By placing the decimal point of the quotient directly above that of the dividend, we locate the decimal point of the quotient automatically in the correct place.

As with multiplication, we need to be careful with zeros in the dividend when locating the decimal point using the method just given. For example, if in the previous example the divisor were 0.4, then we would have

$$0.4\overline{)53.75} \longrightarrow 4\overline{)537.5} \longrightarrow 4\overset{134.375}{\overline{)537.500}} \; .$$

Here the quotient has the same number of decimal places as the dividend only after zeros have been affixed to the dividend.

An alternate explanation to division with decimals that you, the teacher, may appreciate is based upon the equation showing that the product of the quotient and the divisor gives the dividend. It parallels closely the first method shown above.

$$\begin{aligned} 0.5\overset{n}{\overline{)53.75}} \longrightarrow \quad 0.5n &= 53.75 \\ \text{Multiply by } 100: \quad 50n &= 5375 \\ 5(10n) &= 5375 \\ \text{Divide by } 5: \quad 10n &= 1075 \end{aligned}$$

However, this answer, 1075, is ten times as large as we wish. Therefore, the quotient must be $1075 \div 10$; that is, $n = 107.5$.

Exploration of the division process in terms of exponential notation is also revealing. For example, note how we may divide 0.125 by 0.5 :

$$\begin{aligned} 0.125 \div 0.5 &= \frac{125}{10^3} \div \frac{5}{10^1} \\ &= \frac{125}{10^3} \cdot \frac{10^1}{5} \\ &= \frac{125}{5} \cdot \frac{10^1}{10^3} \\ &= 25 \cdot \frac{1}{10^2} \\ &= 0.25 \quad . \end{aligned}$$

Here is a good place to give special attention and emphasis to estimation of answers. This should serve to prevent many errors in location of decimal points. The youngster who recognizes that 21.75 ÷ 5 is approximately 4, will then realize that the decimal point in the quotient should be located as 4.35 .

This completes our discussion on operations with decimals. In the next chapter we will look again at decimals and see how they can be used in developing the set of real numbers.

Class Exercises

12. Find the quotient 0.65 ÷ 2.5 by
    (a) the equation approach as given in this section;
    (b) use of an equivalent problem involving a divisor that is a whole number.

8.3 Ratio and Proportion

A ratio is used to compare two numbers. When we speak of the ratio of two numbers, we are referring to their relative sizes. Thus, if two numbers are in the ratio of 2 to 3, the first is two-thirds as large as the second. If the number of elements in two sets is in the ratio 2 to 3, then every two elements of the first correspond to three elements of the second. This ratio is sometimes written as 2:3 . The ratio indicates a correspondence between the numbers 2 and 3. How many other pairs of numbers have this same correspondence? Some of them are

4 and 6
6 and 9
20 and 30
24 and 36 .

Indeed, we have an unlimited choice of ordered pairs of numbers that have the same correspondence or ratio. In general, any ordered pair of the form $2k$ and $3k$, $\underline{k}$ a counting number, are in the same ratio, 2 to 3. Notice that we refer to these as "ordered" pairs of numbers. Changing the order of the two numbers compared changes their ratio. Thus, while 4 to 6 represents the same ratio as 2 to 3, 6 to 4 does not. The only case where this does not happen is when a number is compared with itself. For example, 40 to 40, 70 to 70, and 257 to 257 all represent the same ratio as 1 to 1.

Since many ordered pairs of numbers may be in the same ratio, it is common to express them in "reduced" form. This requires dividing each number by the greatest common factor. Thus, ratios such as

160 to 4
400 to 10
800 to 20
1600 to 40

are usually written as 40 to 1.

We frequently use ratios, which are comparisons between numbers, when comparing quantities such as distance in miles and time in hours. For example, using the figures above a person who travels 160 miles in 4 hours averages the same rate as one traveling 400 miles in 10 hours or 800 miles in 20 hours. In each case the ratio of the number of miles traveled to the number of hours spent traveling is 40 to 1. Using this ratio, it is likely that each will describe his rate as 40 miles per 1 hour or simply as 40 mph.

In the formula $d = rt$, we find that r equals the ratio of d to t where again the variables represent only numbers. Thus, we write

$$r = \frac{d}{t} .$$

This indicates a fairly common practice of using fractional notation to represent a ratio. In general, the ratio $\underline{a}$ to $\underline{b}$ $(b \neq 0)$ is written either $a:b$ or $\frac{a}{b}$. Every rational number can be thought of as a ratio. This again emphasizes the interpretation of a fraction as an ordered pair of numbers.

If we have two pairs of numbers which represent the same ratio, such as 2 to 3 or 4 to 6, we may write

$$2:3 = 4:6$$

or

$$\frac{2}{3} = \frac{4}{6} .$$

Such a statement of the indicated equality of two ratios is called a proportion.

How can we tell if two ratios are the same? For example, do $\frac{6}{18}$ and $\frac{8}{32}$ represent the same ratio? The ratio of $\frac{6}{18}$ is the same as $\frac{1}{3}$, 1 number in the first set for every 3 in the second. The ratio $\frac{8}{32}$ is the same as $\frac{1}{4}$, 1 member in the first set for every 4 in the second. Clearly these describe different correspondences and we conclude that $\frac{6}{18}$ and $\frac{8}{32}$ do not represent the same ratios.

Treating ratios as rational numbers we can compare them using the property:

$$\frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc.$$

In the example we find $6 \times 32 \neq 18 \times 8$ and thus conclude that the ratios are not equal $(\frac{6}{18} \neq \frac{8}{32})$.

We are often confronted with finding the fourth member in two pairs of numbers which name the same ratio. For example, for what value of d will the ratios $\frac{3}{5}$ and $\frac{6}{d}$ be the same? The problem becomes one of finding a replacement for d that makes the following true:

$$\frac{3}{5} = \frac{6}{d}.$$

The solution can readily be found from the corresponding equation

$$3d = 5 \cdot 6 .$$

You know of the many applications of proportion and we will not dwell on them here. Proportion is also an excellent way to approach percent, as we will do in the next section. As some simple proportion problems, let us solve the following:

1. If the ratio of department head's salary to a teacher's salary is 11 to 10, what increase could a teacher with a salary of \$7600 expect if promoted to department head?

   Solution: Here the proportion is

   $$\frac{11}{10} = \frac{s}{7600} .$$

   Using the condition for equality we have

   $$\begin{aligned} 10 \cdot s &= 11 \cdot 7600 \\ 10s &= 83600 \\ s &= 8360. \end{aligned}$$

   Thus, the increase is \$760.

2. If the ratio of the football coach's salary to a teacher's salary is 115 to 100, what increase could the same teacher expect if assigned as coach?

   Solution:

$$\frac{115}{100} = \frac{s}{7600}$$

$$100s = 115 \cdot 7600$$

$$s = 8740$$

   Thus, the increase is $1140.

The important point here (apart from the salary problem) is the method of solution. It is a general method of solution and is applied in all problem situations of this nature.

Class Exercises

13. You have just finished Chapter 7 in a book of 14 chapters. What is the ratio of the chapters finished to the total number of chapters? What is the ratio of chapters remaining to those finished?

14. Find N in the following proportions:

    (a) $\frac{4}{N} = \frac{12}{15}$ (c) $\frac{N}{5} = \frac{2}{7}$

    (b) $\frac{2}{7} = \frac{N}{49}$ (d) $\frac{6}{2} = \frac{N}{5}$

15. If the ratio of cats to dogs in a certain city is 4 to 3, i.e., $\frac{4}{3}$, how many cats are there to go with the 3663 dogs?

16. In a certain city people are fond of a drink made of two ingredients in a ratio of 3 to 2 (sometimes 3 to 1 or even 4 to 1). How much of the second ingredient should be used to go with 32 ounces of the first ingredient (using the 3 to 2 ratio)?

8.4 Percent

One of the important topics of junior high school mathematics is percent. Yet many students find it either confusing or just plain boring. Part of this reaction has come from the teacher's lack of knowledge concerning the relationship of percent to the rational numbers and mathematics in general. The other part comes from attention to applications of percent valuable to the adult but simply not meaningful or significant to the junior high school student.

The recent approach to the teaching of percent focuses more attention on percent as a way of comparing numbers, as a form of a ratio, as a form of a rational number, while minimizing the usual attention given to applications.

There is nothing mysterious about percents. The word "percent" means hundredths. The symbol "%" is used for convenience and offers a short way of saying "times $\frac{1}{100}$." Thus, 48 percent means

$$48\% = \frac{48}{100} = 48 \cdot \frac{1}{100} .$$

As a ratio 48% compares 48 to 100. Other ratios such as 24 to 50 and 12 to 25 are equal to the ratio 48 to 100 and hence may also be expressed as 48%. While the percent notation is very common and frequently used to express the ratio between two numbers, it is not used directly in calculation. To compute with 48% the percent must first be expressed either as a fraction or as a decimal. To express 48% as a simplified fraction, first write it as a fraction with denominator 100. Then by use of the greatest common factor in numerator and denominator, simplify the fraction:

$$48\% = \frac{48}{100} = \frac{12 \cdot 4}{25 \cdot 4} = \frac{12}{25} \cdot \frac{4}{4} = \frac{12}{25} .$$

To express 48% as a decimal, first express it as a fraction with denominator 100.

$$48\% = \frac{48}{100} = .48$$

To represent a fraction such as $\frac{1}{6}$ as a percent we need to find the number $\underline{c}$ such that

$$\frac{1}{6} = \frac{c}{100} .$$

We know that we can rewrite the above proportion as the equation

$$1 \cdot 100 = 6 \cdot c .$$

Since $6c = 100$, $c = 16\frac{2}{3}$. Thus,

$$\frac{1}{6} = \frac{16\frac{2}{3}}{100} = 16\frac{2}{3}\% .$$

In general, any given ratio, $\frac{a}{b}$, can be expressed as a percent by finding the number $\underline{c}$ such that $\frac{a}{b} = \frac{c}{100}$. The expression $\frac{c}{100}$ means the same as $c \cdot \frac{1}{100}$ or $c\%$.

This development illustrates the role of proportion in percent problems. Many new textbooks including the SMSG Mathematics for Junior High School present percents exclusively through the use of proportions. This generally helps to make the subject clearer to the student. However, students need to master and memorize many of the simpler percent conversions at the junior high level. This requires repeated oral drill and review of the decimal and percent forms of rational numbers like

$$\frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{8},\ \frac{1}{12},\ \frac{1}{20},\ \frac{1}{40}\ .$$

As we insist that students memorize the multiplication table as well as know how to multiply, so should we require the student to memorize basic equivalent forms of percents, decimals, and fractions as well as know how to solve percent problems through proportions.

In the past, many 7th grade books have treated percent in great detail by classifying all problems into three cases. The "three cases" of percent have been overdone and in most new books this treatment is either reduced or not found at all. All "three cases" may be handled the same way, by writing a proportion with one denominator equal to 100 and finding the missing number. Examples follow to illustrate the procedure.

1. If 28 out of 40 transistors failed to meet the specifications for heat sensitivity, what percent of the transistors are defective?

   Solution: Since the total number is 40 and the number defective is 28, the proportion is

   $$\frac{28}{40} = \frac{x}{100}$$

   so that $40x = 2800$,

   $$x = 70\ .$$

   Thus 70% are defective.

2. If Janice needed $70 for a cheerleading outfit and earned 92% of the total by babysitting, how much money did she earn?

   Solution: Here we know both the percent and the total, so that our proportion is

   $$\frac{n}{70} = \frac{92}{100}$$

   and $n = \dfrac{92 \cdot 70}{100}$

   $$n = 64.40\ .$$

   Thus, Janice earned $64.40 .

3. In a box of ballbearings, 11 are rejected by an inspector as being faulty. If this represents 44% of the total production, how many were produced?

Solution: In this case we know the percent and the part or number rejected, so our proportion is

$$\frac{11}{b} = \frac{44}{100}$$

$$44b = 1100$$

$$b = 25 \; .$$

Experienced teachers will recognize each of the above problems as one of the "three cases" of percent. Notice how all three are treated the same way through the use of proportions. With experience students will begin to short-cut writing the proportion, but at the beginning this approach is a simple one for the students to master. Much of the rationale for this work has already been done in the treatment of rationals and decimals, so that most of the material is not new.

One shortcut in writing the proportion is to replace the percent $(\frac{c}{100})$ by an equivalent decimal or simplified fraction. This can be easily done once the students have developed skill in recognizing the relationship among the percents, decimals, and fractional forms of rational numbers. This short-cut replaces the proportion with the familiar formula

$$p = r \times b.$$

The variables p, r, and b represent numbers:

r is the rate or percent;

b is the base or number on which a rate is applied;

p is the percentage or part of the base determined by the rate.

Notice that the term "percentage" has a meaning quite distinct from "percent".

Again percent problems should not be classed into three cases but all solved directly from the same formula, $p = r \times b$. In this form the three examples just given are expressed as follows:

1. $p = r \times b$

   $28 = r \times 40$

   $r = 70\%$

2. $p = r \times b$

   $p = \frac{92}{100} \times 70$

   $p = 64.40$

3. $p = r \times b$

   $11 = \frac{44}{100} \times b$

   $b = 25$

If the student is first introduced to the solving of percent problems through the use of proportions, there is less chance that he will have difficulty identifying the percentage and the base properly when using the equation form.

Class Exercises

17. Write each of the following as a percent:

(a) $\frac{1}{5}$ (c) $\frac{1}{40}$

(b) $\frac{1}{25}$ (d) $\frac{300}{[illegible]}$

18. Find the fraction which corresponds to each of the following percents:

(a) $37\frac{1}{2}$% (c) 26%

(b) $\frac{1}{4}$% (d) 5.125%

19. Changes are often given in terms of percent in order to provide a standard for comparison. Thus, terms such as "an increase of 25%" or "a 10% decrease" are encountered. Students sometimes have difficulty in setting up the appropriate proportion. The change in either case is expressed as a part of the original quantity. Thus, a salary of \$4.40 per hour reflects a 10% increase over a salary of \$4.00 per hour.

If a person receives an 8% pay cut during a "retrenchment" and later receives an 8% increase, how does his final salary compare with his original salary?

20. If a book is printed by a photographic process which reduces the original by 15%, how long should a segment be if it is to be $4\frac{1}{4}$" in the finished book?

Students are frequently troubled with percents less than 1% and greater than 100%. If percents have been introduced as another way of representing a ratio or fractional name for a rational number, then students should have little difficulty with these special percents. Indeed, there is nothing special about them; they carry exactly the same meaning as percents from 1% to 100% and they are used in operations in exactly the same way. Recalling the definition of percent, we can write

$$\frac{1}{2}\% = \frac{1}{2} \cdot 1\% = \frac{1}{2} \cdot \frac{1}{100} = \frac{1}{200} .$$

Similarly,

$$150\% = 150 \cdot 1\% = 150 \cdot \frac{1}{100} = \frac{150}{100} = \frac{3}{2} .$$

Writing a numeral such as $\frac{1}{3}$ or $\frac{1}{8}$ as a percent poses certain problems. Primarily, these are problems of form in notation rather than anything else. Thus, writing $\frac{1}{8}$ as a percent gives

$$\frac{1}{8} = \frac{c}{100}$$

or

$$c = 12\frac{1}{2} .$$

We may write $12\frac{1}{2}\%$, or $\dfrac{12\frac{1}{2}}{100}$, which is awkward and clumsy, or 0.125 . The last form is probably the most useful for calculations.

A similar problem arises when we write $\frac{1}{3}$ as a percent.

$$\frac{1}{3} = \frac{c}{100}$$

or

$$c = 33\frac{1}{3}$$

Here again we have alternate forms for expressing the result. We may write $33\frac{1}{3}\%$, $\dfrac{33\frac{1}{3}}{100}$, $.33\frac{1}{3}$, or $.33\overline{3}$ ... as the resulting percent. More attention will be given to repeating decimals like $.33\overline{3}$ ... in the next chapter.

This chapter has dealt with several topics, some of which appear in any seventh grade book whereas others are found only in newer texts. The discussion of ratio and percent was brief, being related to the previous work on rational numbers and decimals. Presentation of these topics in the classroom is probably best done with the same approach, rather than treating them as completely separate entities that are new in all respects. Using the student's background in these areas makes the topics easier to teach, learn, and recall.

Answers to Class Exercises

1. $(2 \times 10^2) + (7 \times 10) + (4 \times 1) + (5 \times \frac{1}{10}) + (8 \times \frac{1}{10^2})$

2. $(9 \times 1) + (0 \times \frac{1}{10}) + (8 \times \frac{1}{10^2}) + (7 \times \frac{1}{10^3}) + (5 \times \frac{1}{10^4})$

3. $(3 \times 5) + (2 \times 1) + (4 \times \frac{1}{5}) + (1 \times \frac{1}{5^2})$

4. $(2 \times 1) + (0 \times \frac{1}{7}) + (4 \times \frac{1}{7^2}) + (1 \times \frac{1}{7^3}) + (6 \times \frac{1}{7^4})$

5. 0.584 6. 17.92

7. (a) 2^{-7} (b) 2^{-1} (c) 2^{-4} (d) 2^{-3} (e) 2^{-6}

8. (a) $.1_{two}$ (b) $.2_{four}$ (c) $.4_{eight}$ (d) $.5_{ten}$

9. $$\begin{aligned} 0.45 + 0.83 &= (45 \times \frac{1}{100}) + (83 \times \frac{1}{100}) \\ &= (45 + 83) \times \frac{1}{100} \\ &= 128 \times \frac{1}{100} \\ &= \frac{128}{100} \\ &= \frac{100}{100} + \frac{28}{100} \\ &= 1.28 \end{aligned}$$

10. $$\begin{aligned} 0.58 - 0.29 &= (58 \times \frac{1}{100}) - (29 \times \frac{1}{100}) \\ &= (58 - 29) \times \frac{1}{100} \\ &= 29 \times \frac{1}{100} \\ &= \frac{29}{100} \\ &= 0.29 \end{aligned}$$

11. (a) $$\begin{aligned} 0.23 \times 0.45 &= (23 \times \frac{1}{10^2}) \times (45 \times \frac{1}{10^2}) \\ &= (23 \times 45) \times (\frac{1}{10^2} \times \frac{1}{10^2}) \\ &= 1035 \times \frac{1}{10^4} \\ &= 0.1035 \end{aligned}$$

(b) $$\begin{aligned} 0.23 \times 0.45 &= (23 \times 10^{-2}) \times (45 \times 10^{-2}) \\ &= (23 \times 45) \times (10^{-2} \times 10^{-2}) \\ &= 1035 \times 10^{-4} \\ &= 0.1035 \end{aligned}$$

12. (a) $2.5\,\overset{n}{\overline{)0.65}} \longrightarrow 2.5\ n = 0.65$

$250\ n = 65$

$1000\ n = 260$

$n = .26$

(b) $2.5\,\overline{)0.65} \rightarrow 25\,\overline{)6.5} \longrightarrow 25\,\overset{.26}{\overline{)6.50}}$

13. 1 to 2; 1 to 1 .

14. (a) 5 (b) 14 (c) $\frac{25}{7}$ (d) 15

15. 4884

16. $21\frac{1}{3}$ oz.

17. (a) 20% (b) 4% (c) $2\frac{1}{2}$% (d) 5000%

18. (a) $\frac{7}{8}$ (b) $\frac{1}{400}$ (c) $\frac{11}{50}$ (d) $\frac{41}{800}$

19. The salary will drop to 92% of the original salary and then be increased by 8% or an additional 7.36%, to end up as 99.36 of the original. As an example a salary of $1000 would be $920% with an 8% decrease. To increase this salary by 8% we take 8% of $920 or $73.60 so that the final salary is only $993.60 .

20. 5 inches

Chapter Exercises

1. Write each of the following in expanded notation.

   (a) 32.785 (b) 42.341_{five}

2. Which of the following statements are true?

   (a) $3^{-1} = \frac{1}{3}$ (d) $\frac{1}{64} = 2^{-5}$

   (b) $7^0 = 0$ (e) $3^{-3} < 3^{-4}$

   (c) $5^{-3} = \frac{1}{125}$ (f) $4^2 > \frac{1}{4^{-2}}$

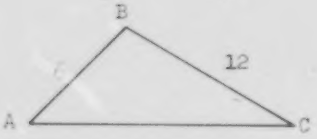
3. Using the fractional approach given in this chapter, evaluate each of the following:

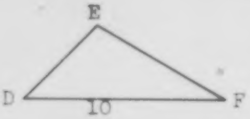
   (a) $0.27 + 0.47$ (b) 0.4×0.37

4. Find N in each of the following proportions:

   (a) $\frac{3}{4} = \frac{N}{7}$ (b) $\frac{5}{6} = \frac{6}{N}$

5. If two triangles have the same shape, we say that they are similar. We define similar triangles to be triangles with corresponding angles congruent and corresponding sides proportional. If $\triangle ABC$ and $\triangle DEF$ are similar with the ratio of corresponding sides 3 to 2, find all sides if $AB = 6$, $BC = 12$, and $DF = 10$.





6. Write each of the following as a percent.

   (a) 10 (b) 1 (c) $\frac{1}{10}$ (d) $\frac{1}{100}$ (e) $\frac{1}{1000}$

7. Let $A = 15$ and $B = 20$.

   (a) A is what percent of B?

   (b) B is what percent of B?

   (c) A is what percent of their sum?

   (d) B is what percent of their product?

   (e) Their difference is what percent of their product?

Chapter 9

The Real Number System

Introduction

This chapter will complete our development of the real number system as it should be seen by the junior high school student. All too frequently, students at this level fail to see the complete picture of the real number system and hence enter into algebra with certain gaps in their background.

9.1 Reviewing Properties of the Rational Number System

In the past chapters we have developed the properties of the rational number system. All solutions to equations of the form $bx = a$, a and b counting numbers, are positive rational numbers. With their opposites (negatives) and zero, they form the complete set of rational numbers.

Likewise, we noted that every rational number can be named by a fraction in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

You already have observed the familiar properties for rational numbers, which may be summarized as follows:

Closure: If a and b are rational numbers, then $a + b$ is a rational number, $a \cdot b$ (more commonly written ab) is a rational number, $a - b$, is a rational number, and $\frac{a}{b}$ is a rational number if $b \neq 0$.

Commutativity: If a and b are rational numbers, then $a + b = b + a$, and $a \cdot b = b \cdot a$, $(ab = ba)$.

Associativity: If a, b, and c are rational numbers, then $a + (b + c) = (a + b) + c$, and $a(bc) = (ab)c$.

Identities: There is a rational number zero such that if a is a rational number, then $a + 0 = 0 + a = a$. There is a rational number 1 such that $a \cdot 1 = 1 \cdot a = a$.

Distributivity: If a, b, and c are rational numbers, then $a(b + c) = ab + ac$.

Additive inverse: If a is a rational number, then there is a rational number (^-a) such that $a + (^-a) = 0$.

Multiplicative inverse: If a is a rational number and $a \neq 0$, then there is a rational number b such that $ab = 1$.

Order: If a and b are different rational numbers, then either $a > b$, or $a < b$.

9.2 Repeating Decimals

Let us look once again at the use of decimals in representing rational numbers. The counting numbers, which form a subset of the set of rational numbers, are expressed in decimal form simply as

$$1, 2, 3, 4, 5, 6, 7, \ldots$$

Other positive rational numbers written in fractional form can readily be represented as decimals. If a fraction has a denominator that is a power of ten, it is easy to write the fraction as a decimal because our decimal system of notation is based on powers of ten. For example:

$$\frac{3}{10} = 0.3 \qquad \frac{37}{100} = 0.37 \qquad \frac{253}{1000} = 0.253$$

If the denominator of a fraction is not a power of ten, the fraction can often be changed to an equivalent one whose denominator is a power of ten. For example:

$$\frac{3}{5} = \frac{6}{10} = 0.6 \qquad \frac{1}{8} = \frac{125}{1000} = 0.125$$

On the other hand, a fraction like $\frac{1}{7}$ cannot be written with a denominator that is a power of ten and a numerator that is a counting number. To show that this cannot be done, suppose for a moment we _assume_ that we can write such an equivalent fraction. If such a fraction does exist, then we would have two ways of naming the number one-seventh and we could write

$$\frac{1}{7} = \frac{a}{10^n}$$

where _a_ is a counting number and _n_ indicates the power of ten. Using the property that if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$, we get

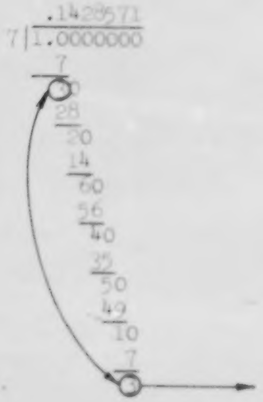
$$1 \cdot 10^n = 7 \cdot a \quad .$$

Now 10 can be factored as $2 \cdot 5$, so $10^n = (2 \cdot 5)^n = 2^n \cdot 5^n$. Thus we can write

$$1 \cdot 2^n \cdot 5^n = 7 \cdot a \quad .$$

The expressions on the left and the right of the equation represent two factorizations of the same number. One involves the prime factor 7; the other does not. But this is impossible since the Fundamental Theorem of Arithmetic says that a number has exactly one unique prime factorization. Therefore, we conclude that our original assumption is false, and that $\frac{1}{7}$ cannot be expressed as a fraction with a denominator that is a power of ten and a numerator that is a whole number.

We can, of course, express $\frac{1}{7}$ in decimal form by dividing the numerator 1 by the denominator 7. We already know that at no stage in the division can we have a 0 remainder because this implies that we can write $\frac{1}{7}$ as a fraction with a denominator that is a power of 10. Therefore, as we divide, there are only six remainders possible, (1, 2, 3, 4, 5, 6). As soon as one of these numbers appears for a second time, the sequence of digits in the quotient will repeat.



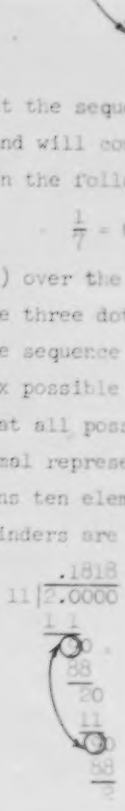
This is the same as the first remainder.

At this point the sequence of digits 142857 begins to repeat itself in the quotient and will continue to repeat indefinitely. The quotient is usually written in the following form.

$$\frac{1}{7} = 0.142857\overline{142857} \ \ldots$$

The bar (vinculum) over the sequence 142857 indicates the set of digits that repeats. The three dots indicate that the pattern repeats indefinitely.

Note that the sequence of digits started to repeat in the quotient as soon as one of six possible remainders appeared for the second time. This does not imply that all possible remainders must appear. Consider, for example, the decimal representation for $\frac{2}{11}$. Here the set of possible remainders contains ten elements; however, the repetition begins after only two of these remainders are used.



$$\frac{2}{11} = 0.18\overline{18} \ \ldots$$

Other examples of this notation for repeating decimals are given here.

$$\frac{1}{9} = .11\overline{1}\ \ldots\ , \quad \frac{3}{13} = .230769\overline{230769}\ \ldots\ , \quad \frac{1}{999} = .001\overline{001}\ \ldots$$

The symbolism adopted for repeating decimals can be used for all decimal expansions of rational numbers. For example, we may write

$$\frac{3}{5} = 0.6 = 0.60\overline{0}\ \ldots$$

$$\frac{1}{8} = 0.125 = 0.1250\overline{0}\ \ldots$$

In this sense we can then say that every rational number can be expressed as a repeating decimal, often called a periodic decimal. Some of these, as $\frac{3}{5}$ and $\frac{1}{8}$ above, will repeat only zeros. These are frequently called "terminating" in the sense that the repeating zero need not be written in the decimal.

How can we tell when a fraction $\frac{p}{q}$ can be written as a fraction with a denominator that is a power of ten? These are the fractions that have terminating decimal forms, that repeat zeros only. We start with the rational number $\frac{p}{q}$, p and q relatively prime. That is, let $\frac{p}{q}$ be in lowest terms.

First let us note, intuitively, that if a fraction has a denominator that can be written as the product of a power of 2 and/or a power of 5, then it can be expressed as a power of 10. Here are some examples:

1. $\frac{13}{40} = \frac{13}{2^3 \cdot 5} \cdot \frac{5^2}{5^2} = \frac{13 \cdot 5^2}{2^3 \cdot 5^3} = \frac{325}{(2 \cdot 5)^3} = \frac{325}{10^3} = \frac{325}{1000}$

2. $\frac{173}{2500} = \frac{173}{2^2 \cdot 5^4} \cdot \frac{2^2}{2^2} = \frac{692}{2^4 \cdot 5^4} = \frac{692}{(2 \cdot 5)^4} = \frac{692}{10^4} = \frac{692}{10000}$

In other words, we can multiply by appropriate powers of 2 and 5 in order to produce a denominator that is a power of 10.

In general, we wish to see which rational numbers $\frac{p}{q}$ can be written in the form $\frac{N}{10^K}$, where N and K are counting numbers. Let us assume we have the following:

$$\frac{p}{q} = \frac{N}{10^K}\ .$$

Therefore, $$q \cdot N = p \cdot 10^K$$

and $$N = \frac{p \cdot 10^K}{q}$$

Now since p and q are relatively prime, q does not divide p, and must therefore divide 10^K. But the only possible factors of 10^K are numbers that are powers of 2 or 5. Thus we may conclude that the rational number $\frac{p}{q}$ can be written as a fraction with denominator that is a power of 10 if and only if the denominator q can be expressed in the form $q = 2^m \cdot 5^n$, m and n whole numbers.

Class Exercises

1. Give the next five digits in each of the following decimal expressions.
   (a) $.27\overline{27}\ldots$ (c) $.1133\overline{1331}\ldots$
   (b) $.4125\overline{4125}\ldots$ (d) $.121\overline{13}\ldots$
2. Which of the following are true statements?
   (a) $.3737\overline{37}\ldots = .37\overline{37}\ldots$
   (b) $.373\overline{73}\ldots < .37\overline{37}\ldots$
   (c) $.377\overline{377}\ldots = .37737\overline{7}\ldots$
   (d) $.37\overline{37}\ldots > .377\overline{377}\ldots$
3. Using the notation of this section, express each of the following in decimal form.
   (a) $\frac{1}{3}$ (b) $\frac{2}{7}$ (c) $\frac{1}{99}$ (d) $\frac{2}{13}$
4. Which of the following fractions can be expressed as "terminating" decimals?
   (a) $\frac{7}{60}$ (b) $\frac{11}{80}$ (c) $\frac{101}{300}$ (d) $\frac{399}{400}$

We have seen that every rational number can be written as a repeating decimal. A related question is whether every repeating decimal names a rational number. Certainly, there is no problem if the decimal expansion "terminates" (repeats zeros). For example:

$$0.23 = \frac{23}{100}, \qquad 0.7156 = \frac{7156}{10000}$$

For decimals that have sequences of repeating digits, not all zero, other methods are needed.

One method for expressing a repeating decimal in fraction form uses the clever technique of subtracting out the repeating digits of a non-terminating decimal thereby producing a terminating decimal that can easily be handled. This manipulative "trick" is illustrated in the examples that follow.

Let $N = .4545\overline{45} \ldots$

First multiply N by 100, and then subtract N from the product.

$$\begin{aligned} 100\,N &= 45.45\overline{45} \ldots \\ -\;N &= .45\overline{45} \ldots \\ \hline 99\,N &= 45.000\overline{0} \ldots \\ 99\,N &= 45 \\ N &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

Note that multiplying N by 100 has the effect of aligning the repeating sequences of the decimals in N and $100\,N$ so that they can be subtracted to give zero in each case. The example shows that

$$.4545\overline{45} = \frac{5}{11} .$$

As another example, let

$$N = .123\overline{123} \ldots$$

First multiply by 1000, and then subtract as before. This gives a new decimal where the repeating sequences are zeros. Do you see why 1000 was chosen for the multiplier here?

$$\begin{aligned} 1000\;N &= 123.123\overline{123} \ldots \\ -\;N &= .123\overline{123} \ldots \\ \hline 999\;N &= 123 \\ N &= \frac{123}{999} \\ &= \frac{41}{333} \end{aligned}$$

Notice that the repeating zeros found by subtraction have not been written in this solution. From the example we see that

$$.123\overline{123} \ldots = \frac{41}{333} .$$

Finally consider $N = 2.4756\overline{56} \ldots$

$$\begin{aligned} 100\,N &= 247.5656\overline{56} \ldots \\ -\;N &= 2.4756\overline{56} \ldots \\ \hline 99\,N &= 245.09 \\ N &= \frac{245.09}{99} \end{aligned}$$

This can also be written as $\frac{24509}{9900}$, which is clearly a rational number. That is, we have shown that $2.4756\overline{56}$ ... is the name of a rational number.

The method just described is found in many junior high school mathematics texts. While appearing plausible at first glance, a closer study should reveal that a very fundamental assumption underlies the technique. Indeed, can we really multiply and subtract "infinite" decimals in this manner at all? We would like to say (actually we assume) the answer is yes.

Other methods are available for expressing repeating decimals in fraction form. They too assume certain properties regarding computing with "infinite" decimals. One involves the use of decimal forms of unit fractions with denominators one less than successive powers of ten. For example:

$$\frac{1}{9} = .111\overline{1}\ \ldots$$
$$\frac{1}{99} = .0101\overline{01}\ \ldots$$
$$\frac{1}{999} = .001001\overline{001}\ \ldots$$
$$\frac{1}{9999} = .0001\overline{0001}\ \ldots$$

The pattern in these decimals should be apparent.

To write $.45\overline{45}\ \ldots$ as a fraction we proceed as follows:

$$\begin{aligned} .45\overline{45}\ \ldots &= 45(.01\overline{01}\ \ldots) \\ &= 45 \cdot \frac{1}{99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

The technique will work for any number of digits in the repeating sequence of the decimal. For example:

$$\begin{aligned} .123\overline{123}\ \ldots &= 123\ (.001\overline{001}\ \ldots) \\ &= 123 \cdot \frac{1}{999} \\ &= \frac{123}{999} \\ &= \frac{41}{333} \end{aligned}$$

This method treats repeating decimals essentially as infinite geometric series. This, of course, is exactly what they are.

$$.45\overline{45}\ \ldots = \frac{45}{100} + \frac{45}{100^2} + \frac{45}{100^3} + \ \ldots$$

$$.123\overline{123}\ \ldots = \frac{123}{1000} + \frac{123}{1000^2} + \frac{123}{1000^3} + \ \ldots$$

The examples of this section suggest the following conclusion that we shall accept as true.

> Every rational number can be expressed as
> a repeating decimal, and every repeating
> decimal names a rational number.

Class Exercises

5. Write the following products as repeating decimals if $N = .29\overline{29} \ldots$

   (a) $10\ N$　　　　(c) $2\ N$

   (b) $100\ N$　　　　(d) $7\ N$

6. Write the following differences as repeating decimals if $A = .22\overline{2} \ldots$ and $B = .33\overline{3} \ldots$

   (a) $B - A$　　　　(b) $10A - B$

7. Express each of the following as a rational number in fractional form.

   (a) $.27\overline{27} \ldots$　　　　(b) $.135\overline{135} \ldots$

8. (a) Express each of the following as a fraction.

   $.99\overline{9} \ldots$　　　　$.49\overline{9} \ldots$

   (b) On the basis of the results found in part (a), is it true that

   $$.99\overline{9} \ldots = 1.00\overline{0} \ldots$$

   and

   $$.49\overline{9} \ldots = .50\overline{0} \ldots \quad ?$$

   (c) Does every terminating decimal have a second corresponding decimal form that is non-terminating?

9.3 Irrational Numbers

In our discussion of the positive rational numbers we noted that they could be defined as the solutions to equations of the form

$$bx = a$$

where a and b are counting numbers. Thus

$$2x = 6, \quad 5x = 9, \quad 7x = 4$$

all have solutions that are positive rational numbers.

Let us turn our attention to another form of equation. What is the nature of the solutions to equations of the form

$$x^2 = k$$

where k is a counting number?

What values of x make the sentences

$$x^2 = 2, \qquad x^2 = 5, \qquad x^2 = 7$$

true? With these questions we open up an entirely new area of investigation.

Can $x^2 = 2$ be solved with a rational number? That is, is there a rational number $\frac{a}{b}$ such that

$$\left(\frac{a}{b}\right)^2 = 2 \quad ?$$

Before answering this question, let us consider some appropriate remarks from number theory.

Let p be a whole number. The number 2p is then an even number. Likewise, its square $(2p)^2 = 4p^2$ is an even number. For any whole number q, $2q + 1$ is an odd number. Similarly, its square,

$$(2q + 1)^2 = 4q^2 + 4q + 1$$

is an odd number. Thus, we have established the properties that:

(1) If a number is even, then the square of the number is even.

(2) If a number is odd, then the square of the number is odd.

Likewise, we can establish two additional properties:

(3) If the square of a number is odd (not even), then the number is odd (not even).

(4) If the square of a number is even (not odd), then the number is even (not odd).

(Those readers familiar with logic will note that the last two properties are the contrapositives of the first two and hence must necessarily be true.)

We will now use these four properties to investigate the nature of the solution to the equation $x^2 = 2$.

Let us assume that $x^2 = 2$ has the rational number $\frac{a}{b}$ as a solution. Further, let us assume that $\frac{a}{b}$ is in reduced form, where a and b are relatively prime. This occurs when the greatest common factor of a and b is 1. In other words, a and b have no factors in common other than 1. If a solution in the form $\frac{c}{d}$ is found for the equation $x^2 = 2$, then an equivalent fraction $\frac{a}{b}$ (a and b relatively prime) always exists. Thus, assuming that $x^2 = 2$ has the solution $x = \frac{a}{b}$, we get

$$(\frac{a}{b})^2 = 2$$

$$\frac{a^2}{b^2} = 2$$

$$a^2 = 2b^2$$

We see that a^2 is even, for $a^2 = 2b^2$ or a^2 is of the form $2n$. From property (4) just mentioned, this means that *a is even* also.

Since *a* is even, we may write $a = 2c$, *c* a counting number. With this in mind we may also write $a^2 = 2b^2$ as

$$(2c)^2 = 2b^2$$

$$4c^2 = 2b^2$$

$$2c^2 = b^2$$

This leads us to the conclusion that since $2c^2$ is even, b^2 must be even. However, if b^2 is even, *b is even*.

From our *assumption* that $x^2 = 2$ has the solution $\frac{a}{b}$ (*a* and *b* having no factors in common), we are forced to conclude that both *a* and *b* are even. But this is impossible, since for *a* and *b* to be even, both must have the common factor 2. Thus, we can only conclude that our assumption was false; the solution to $x^2 = 2$ *cannot* be written in the form $\frac{a}{b}$ and is not a rational number.

We need a number other than those in the set of rational numbers for a solution to $x^2 = 2$. We agree to write one solution as $\sqrt{2}$ such that the number $\sqrt{2}$ has the property that

$$\sqrt{2} \cdot \sqrt{2} = (\sqrt{2})^2 = 2 .$$

(From our study of negative numbers, we see that ${}^-\sqrt{2}$ is also a solution to $x^2 = 2$ since $({}^-\sqrt{2}) \cdot ({}^-\sqrt{2}) = ({}^-\sqrt{2})^2 = 2$.)

The number $\sqrt{2}$ is an example of an irrational number. Others are

$$\sqrt{3}, \quad \sqrt{5}, \quad \sqrt{7}$$

which you may recognize as solutions to

$$x^2 = 3, \quad x^2 = 5, \quad x^2 = 7 .$$

While these are irrational numbers, we can make rational approximations to them.

For example, to make a rational number approximation to $\sqrt{2}$, we proceed as follows.

$$1 < \sqrt{2} < 2$$

$$1.4 < \sqrt{2} < 1.5$$

$$\text{since } (1.4)^2 = 1.96 \text{ and } (1.5)^2 = 2.25$$

$$1.41 < \sqrt{2} < 1.42$$

$$\text{since } (1.41)^2 = 1.9881 \text{ and } (1.42)^2 = 2.0164$$

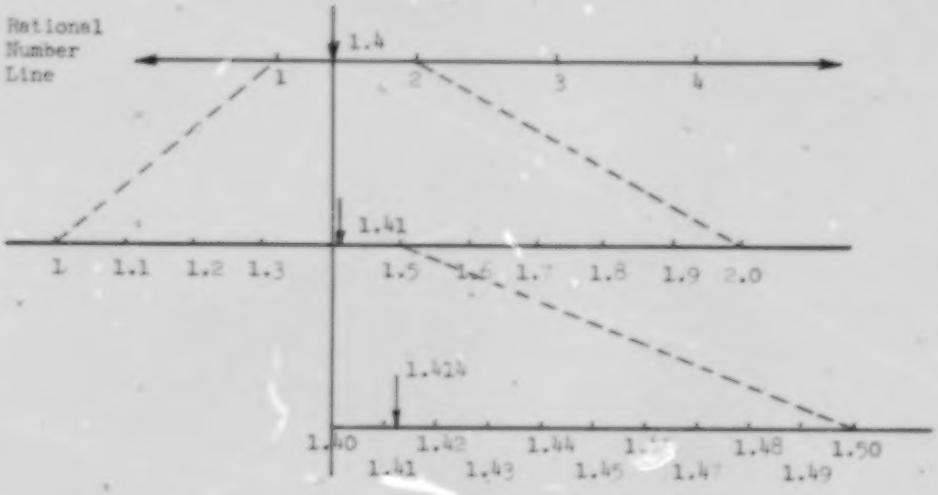
$$1.414 < \sqrt{2} < 1.415$$

Continuing this process we can approximate the value of the irrational number $\sqrt{2}$ between two rational numbers to the nearest ten-millionth, as

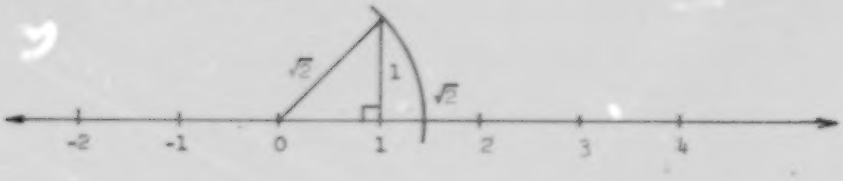
$$1.4142135 < \sqrt{2} < 1.4142136 \quad .$$

Since $(1.4142135)^2 = 1.99999982358225$, we have a rather good approximation to $\sqrt{2}$.

Pictorially, we can represent this approximating process on the rational number line by enlarging sections successively, as needed, to show the finer subdivisions.


Rational Number Line
1.4
1
2
3
4
1.41
1
1.1
1.2
1.3
1.5
1.6
1.7
1.8
1.9
2.0
1.414
1.40
1.42
1.44
1.46
1.48
1.50
1.41
1.43
1.45
1.47
1.49

We note at this point that while $\sqrt{2}$ is an irrational number, it does correspond to a particular point on the number line if we think of the line as a continuous set of points with no gaps. This is readily illustrated using the Pythagorean property.


$\sqrt{2}$
1
$\sqrt{2}$
-2
-1
0
1
2
3
4

Apparently there are points on the line that do not correspond to rational numbers. This is indeed true. In fact there are infinitely many points on the number line that cannot be named with rational numbers. Each corresponds to an irrational number.

Class Exercises

9. Every counting number n is either even or odd. If n is even, we may find a whole number p such that

$$n = 2p .$$

If n is odd, we may find a whole number q such that

$$n = 2q + 1 .$$

(a) For each n listed in the set { 2, 6, 18, 48 }, find an appropriate p.

(b) For each n listed in the set { 3, 7, 33, 99 }, find an appropriate q.

10. By the method of squaring shown in the text, verify that $1.732 < \sqrt{3} < 1.733$.

In the SMSG Introduction to Secondary School Mathematics, Volume 2, there appears a proof that the solution to $x^2 = 2$ is not rational which is based upon the possible cases of oddness and evenness for a and b; namely, (1) a even, b even: (2) a even, b odd: (3) a odd, b even: and (4) a odd, b odd.

Another interesting proof may be shown with base three numerals. Base three numerals end only in the digits 0, 1, or 2. If they end in 0, their squares end in 00. If they end in 1, their squares end in 1. If they end in 2, their squares end in 1. Hence the squares of any number (except 0) written in base three ends in 00 or 1. The extended multiplication table for base

Extended Multiplication Table
Base Three

x	0	1	2	10	11	12	20	21	22	100
0	0									
1		1								
2			11							
10				100						
11					121					
12						221				
20							1100			
21								1211		
22									2101	
100										10000

three illustrates this property.

As before, we assume that $x^2 = 2$ has a solution $x = \frac{a}{b}$ (a and b relatively prime) and therefore $a^2 = 2b^2$. First, consider the case when the base three numeral for b^2 ends in 1. From $a^2 = 2b^2$ we see that the base three numeral for a^2 must end in 2.

$$a^2 = 2b^2 = 2(\_\,\_\,\_\,1) = \_\,\_\,\_\,2$$

But this is impossible since no squares have base three numerals ending in the digit 2. Hence the assumption that the numeral for b^2 ends in 1 is false. We consider the only other possible case: the numeral for b^2 ends in the digits 00. This time because $a^2 = 2b^2$ we see that the numeral for a^2 must end in the digits 00.

$$a^2 = 2b^2 = 2(\_\,\_\,0\,0) = \_\,\_\,0\,0$$

Now if the numeral for b^2 ends in 00, the numeral for b ends in 0. Likewise, the numeral for a^2 ends in 00 and hence the numeral for a ends in 0. But every base three numeral ending in 0 is divisible by three (10_{three}). Hence b and a both are divisible by three; they have a common factor. However, this is a contradiction since part of our original assumption was that a and b are relatively prime.

Our only conclusion can be that our original assumption that $x^2 = 2$ has the solution $x = \frac{a}{b}$ (a and b relatively prime) is not true. The solution to $x^2 = 2$ is not a rational number.

In this section we have shown that $x^2 = 2$ cannot be solved by a rational number. We gave one solution the name $\sqrt{2}$ and called it an irrational number. (Recall a second solution to the same equation is $^-\sqrt{2}$ which is also irrational.) Other similar sentences have irrational numbers such as $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ for solutions. There are many other types of irrational numbers. One very familiar irrational number is π. When we use $\frac{22}{7}$ or 3.14 for π in our computations, we are only using rational number approximations to the irrational number π. Still other examples of irrational numbers are given here.

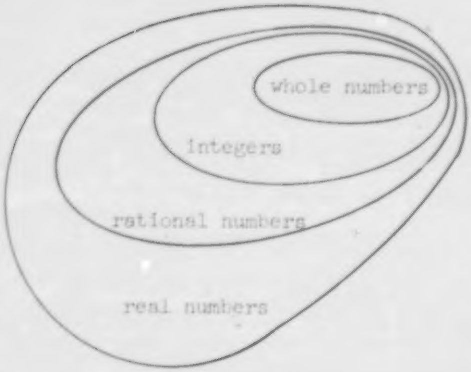
$$2\pi, \quad -\pi, \quad 2 + \sqrt{2}, \quad \sqrt{2} + \sqrt{3}, \quad 2\sqrt{2}, \quad (\sqrt{2})^3$$

In the next section we will learn more about these numbers that are not rational.

9.4 Real Numbers

We have learned that various kinds of questions may be asked with counting numbers. Many of these take the form of open number sentences such as $5x = 75$ that can be solved with counting numbers. Others, such as $3x = 2$ are answered with positive rational numbers. Still others, like $x^2 = 3$, can be answered only with irrational numbers.

The seventh grader should be familiar with the counting numbers and the positive rational numbers. In the eighth grade he will learn about the irrational numbers and will study negative as well as positive numbers. Attention will be placed upon these numbers and their properties and how they develop into the set of real numbers.



In the SMSG text Mathematics for Junior High School, Volume II, a chapter is devoted to the real number system. We will give here some of the key ideas presented in the chapter not because they belong in the seventh grade, but because the seventh grade teacher should have this background. One of the primary objectives of the junior high school mathematics program is to show students the developing number system from the basic counting numbers to the real number system. Granted, this can only be done informally at this level. Yet it is essential that students begin to see the relation between the completeness of the real numbers and the continuity of points on the real number line.

We approach our study of the real numbers through the use of decimals. Seventh grade youngsters are often hasty to assume that all decimal representations are names of rational numbers. Such, of course, is not the case. We need to explore the set of decimal expansions that do not repeat; that is, that are not periodic.

We have already encountered an example of an irrational number in an earlier section when we discussed $\sqrt{2}$. Although $\sqrt{2}$ is a bona-fide number, it has a decimal expansion that is non-periodic. How do we know that the decimal expansion for $\sqrt{2}$ does not repeat? It has an infinite number of digits; we could never hope to visually check to see that it has no repeating sequences! Recall that we exhibited a method for representing every repeating decimal in fractional form and concluded that all repeating decimals named rational numbers. Now $\sqrt{2}$ cannot be expressed in fractional form as we have already shown. Hence, it cannot be expressed as a repeating decimal. That is, the decimal expansion for $\sqrt{2}$ does not repeat; it does not terminate. Indeed this is what distinguishes it from the class of numbers that we have been discussing thus far. We now can define rational and irrational numbers in terms of their decimal representations.

> A rational number is any number that has a periodic (repeating) decimal representation.
>
> An irrational number is any number that has a non-periodic (non-repeating) decimal representation.

The system composed of both the rational and irrational numbers is the real number system. Every real number is either rational or irrational. If the decimal representation is periodic, the number is a rational number; otherwise, the number is an irrational number.

Each of the following is the name of a real number. Can you tell which ones represent rational numbers?

a. $0.123\overline{123}\ldots$
b. 0.2578
c. $0.3720\overline{0}\ldots$
d. $0.10110111011110\ldots$
e. $0.213213321333\ldots$

The first three decimals on the list are names for rational numbers; they are periodic decimals. (Recall that a decimal such as 0.2578 can be thought of as repeating zeros thereafter.) The last two are obviously not periodic and therefore represent irrational numbers. Both have a pattern to show you how to continue to write additional numerals in the sequence, but this pattern does not consist of a set of repeating digits. All five decimals, however, are representations of real numbers.

A detailed study of the decimal representation of the set of real numbers, together with the properties of the real number system, does not normally occur until the eighth grade course. It has been included here in order to provide you with a brief overview of the development of our number system. The set of real numbers is now said to be complete. Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.

One should not infer from the above illustrations that all irrational numbers have decimal representations which, while non-repeating, do exhibit patterns. The digits in the decimal expansion for $\sqrt{2}$ have no pattern.

$$\sqrt{2} = 1.4142135\ \ldots$$

Likewise, the decimal expansion for π at no point exhibits anything other than random ordering of digits.

$$\pi = 3.\ 14159\ 26534\ 89793\ 23846\ 26433\ 83279\ \ldots$$

It is with respect to this latter point that the set of rational numbers differs from the set of real numbers. For each rational number there corresponds a point on the number line, but there are points on the number line that do not correspond to any rational number. For example, $\sqrt{2}$ is not a rational number, yet can be located on the number line.

The set of real numbers, as well as the sets of rational and irrational numbers, are said to be dense. That is, between any two distinct real numbers there is always another real number. Indeed, between any two real numbers there are infinitely many more real numbers. For example, consider the real numbers:

a. $2.345345\overline{345}\ \ldots$ (rational)
b. $2.345534555\ \ldots$ (irrational)

To locate a real number between these two we need to have in the fourth decimal place a digit between 3 and 5, that is, 4. Thereafter, by our pattern, we can locate either a rational or an irrational number between the two given numbers. Here is an example of each:

a. $2.3453\ 45\overline{345}\ \ldots$
rational : $2.3454\ \overline{3454}\ \ldots$
irrational : $2.3454\ 5445444\ \ldots$
b. $2.3455\ 34555\ \ldots$

Can you find others?

Class Exercises

11. Classify each of the following as rational or irrational.
    - (a) $0.185\overline{185}\ldots$
    - (b) $0.070770777\ldots$
    - (c) $0.112111221111222\ldots$
    - (d) 3.1416
    - (e) $\sqrt{25}$
    - (f) $4.250\overline{0}\ldots$

12. Write the next nine digits in the decimal expansion of the real numbers given in parts (a), (b), and (c) of the preceding exercise.

13. Write a decimal for a rational number between $2.384\overline{384}\ldots$ and $2.385\overline{385}\ldots$

14. Write a decimal for an irrational number between $0.7254\overline{7254}\ldots$ and $0.7255\overline{7255}\ldots$

15. Order the following real numbers from smallest to largest: $.3434\overline{34}\ldots$, $.343443444\ldots$, $.344\overline{344}\ldots$, $.343343334\ldots$, $\frac{17}{50}$, $\frac{1}{3}$, $\frac{172}{500}$.

9.5 Properties of the Real Number System

We have presented, in Chapters 5-9, a development of the properties of number systems from the set of counting numbers through the set of real numbers. This material is normally developed in far more detail than given here, as part of the mathematics program of grades 7 and 8. It is important to have youngsters see the overall structure of the real number system, together with the properties of the various subsets of the set of real numbers. It is equally important, however, that opportunities be provided for practice of computational skills at this grade level. Neither of these aspects should be neglected.

In summary of these last chapters we present here the properties of the real number system.

Property 1. Closure

(a) Closure under Addition. The real number system is closed under the operation of addition. If a and b are real numbers then $a + b$ is a real number.

(b) Closure under Subtraction. The real number system is closed under the operation of subtraction (the inverse of addition). If a and b are real numbers then $a - b$ is a real number.

(c) Closure under Multiplication. The real number system is closed under the operation of multiplication. If a and b are real numbers then $a \cdot b$ is a real number.

(d) Closure under Division. The real number system is closed under the operation of division (the inverse of multiplication). If a and b are real numbers then $a \div b$ (when $b \neq 0$) is a real number.

The operations of addition, subtraction, multiplication, and division on real numbers display the properties which we have already observed for rationals. These may be summarized as follows:

Property 2. Commutativity

(a) If a and b are real numbers, then $a + b = b + a$.

(b) If a and b are real numbers, then $a \cdot b = b \cdot a$.

Property 3. Associativity

(a) If a, b, and c are real numbers, then $a + (b + c) = (a + b) + c$.

(b) If a, b, and c are real numbers, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Property 4. Identities

(a) If a is a real number, then $a + 0 = 0 + a = a$. Zero is the identity element for the operation of addition.

(b) If a is a real number, then $a \cdot 1 = 1 \cdot a = a$. One is the identity element for the operation of multiplication.

Property 5. Distributivity

If a, b, and c are real numbers, then $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Property 6. Inverses

(a) If a is a real number, there is a real number (^{-}a), called the additive inverse of a such that $a + (^{-}a) = 0$.

(b) If a is a real number and $a \neq 0$ there is a real number b, called the multiplicative inverse of a such that $a \cdot b = 1$.

Property 7. Order

The real number system is ordered. If a and b are different real numbers then either $a < b$ or $a > b$.

Property 8. Density

The real number system is everywhere dense. Between any two distinct real numbers there is always another real number. Consequently, between any two real numbers we find as many more real numbers as we wish. In fact we easily see that: (1) There is always a rational number between any two distinct real numbers, no matter how close; (2) There is always an irrational number between any two distinct real numbers, no matter how close.

The ninth property of the system of real numbers is one which is not shared by the rationals.

Property 9. Completeness

The real number system is complete. Not only does a point on the number line correspond to each real number, but conversely, a real number corresponds to each point on the number line.

Answers to Class Exercises

1. (a) 27272 (b) 41254 (c) 33133 (d) 13131

2. a

3. (a) $0.33\overline{3}\ldots$ (c) $.0101\overline{01}\ldots$
   (b) $0.714285\overline{714285}\ldots$ (d) $.153846\overline{153846}\ldots$

4. b, d

5. (a) $2.929\overline{29}\ldots$ (c) $.58\overline{58}\ldots$
   (b) $29.29\overline{29}\ldots$ (d) $2.05\overline{05}\ldots$

6. (a) $.11\overline{1}\ldots$ (b) $1.88\overline{8}\ldots$

7. (a) $\frac{3}{11}$ (b) $\frac{5}{37}$

8. (a) $\frac{1}{1}$, $\frac{1}{2}$ (b) yes (c) yes

9. (a) 1, 3, 9, 24 (b) 1, 3, 16, 29

10.
$$\begin{array}{ccccc} 1.732 & < & \sqrt{3} & < & 1.733 \\ (1.732)^2 & < & (\sqrt{3})^2 & < & (1.733)^2 \\ 2.999824 & < & 3 & < & 3.003289 \end{array}$$

11. (a) rational (d) rational
    (b) irrational (e) rational
    (c) irrational (f) rational

12. (a) 185185185 (b) 077770777 (c) 111112222

13. Answers will vary: two possible answers are
    $2.3847\overline{3847}\ldots$, 2.38473845

14. Answers will vary: two possible answers are
    0.725450550 ... , 0.7254854885488 ...

15. $\frac{1}{3}$, $\frac{17}{50}$, .343343334 ... , $.3434\overline{34}\ldots$, .343443444 ... ,
    $\frac{172}{500}$, $.344\overline{344}\ldots$

Chapter Exercises

1. Write the decimal expansion for each of the following rational numbers:
(a) $\frac{2}{3}$ (b) $\frac{4}{9}$ (c) $\frac{3}{11}$ (d) $\frac{2}{99}$

2. Give the next five digits in each of the following decimal expansions:
(a) $.35\overline{35}\ldots$ (b) $.353553555\ldots$ (c) $.355\overline{355}\ldots$

3. Write a rational number in fractional form for each of the following:
(a) $0.12\overline{12}\ldots$ (b) $0.432\overline{432}\ldots$ (c) $.699\overline{9}\ldots$

4. Classify each of the following as either a rational or an irrational number.
(a) $\sqrt{5}$ (b) $(\sqrt{5})^2$ (c) $^-\sqrt{3}$ (d) $\frac{\pi}{2}$ (e) $\sqrt{2} - \sqrt{2}$

5. Repeat Exercise 4 for the following:
(a) $.17\overline{17}\ldots$ (b) $.171771777\ldots$ (c) $.17117$
(d) $.171171117\ldots$ (e) $.1700\overline{0}\ldots$

6. Which of the following numbers is the largest? Which is the smallest?
(a) $.43$ (b) $.43\overline{43}\ldots$ (c) $.434334333\ldots$
(d) $.434\overline{34}\ldots$ (e) $.4344344\ldots$

7. Write two decimals for (a) a rational number and (b) an irrational number between $0.345\overline{345}\ldots$ and $0.345334533345\ldots$

8. Write the decimal expansions for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$.
Try to find a pattern that recurs in each of these representations.

9. Repeat Exercise 8 for the thirteenths from $\frac{1}{13}$ through $\frac{12}{13}$.

10. Between what two consecutive counting numbers are the following?
(a) ______ $< \sqrt{3} <$ ______ (e) ______ $< \sqrt{5} <$ ______
(b) ______ $< \sqrt{123} <$ ______ (f) ______ $< \sqrt{91} <$ ______
(c) ______ $< \sqrt{11} <$ ______ (g) ______ $< \sqrt{224} <$ ______
(d) ______ $< \sqrt{29} <$ ______ (h) ______ $< \sqrt{69} <$ ______

11. Solve each equation.

(a) $3x = 91$

(b) $x^2 = 3$

(c) $4x = 96$

(d) $28x = 1$

(e) $15x = 75$

(f) $14x = 70$

(g) $8x = 63$

(h) $x^2 = 11$

(i) $31x = 558$

(j) $49x = 98$

(k) $7x = 231$

(l) $8x^2 = 232$

(m) $11x = 176$

(n) $11x = 175$

(o) $x^2 = 13$

(p) $x^2 = 5$

Classify each solution as a counting number, a rational number (not a counting number), or an irrational number.

Chapter 10

NON-METRIC GEOMETRY, I

More and more of the basic concepts of geometry are being introduced at the junior high level or even earlier. This is not to say geometry is being treated as a deductive system in grade seven but that these students are learning many of the fundamental ideas of geometry. Several reasons exist for this increasing emphasis on geometry. Many topics in mathematics are being introduced earlier than previously was the case; geometry is one of them. The demise of solid geometry as a full course in its own right and the inclusion of much of its content in the tenth grade geometry course has made it difficult to introduce all the three-dimensional concepts and study them in any depth in the time allotted. The study of geometry introduces a new element into the junior high school years which in the past have been primarily concerned with arithmetic. Junior high school students enjoy geometry and easily learn many of the concepts which have a "pay-off" in the future.

Seventh and eighth grade books today, including the SMSG Mathematics for Junior High School, Volumes I and II, include many topics usually not encountered until grade ten. They study the relationships between points, lines, and planes in space; angles, triangles, and polygons; parallels and parallelograms; basic ideas of measure and congruence; as well as properties of solids.

This chapter, and the next, will treat many of those aspects of geometry which do not depend upon the concept of distance or measurement. Chapters 12 and 13 will introduce the idea of measure and use it to enlarge the study of geometry.

You are aware that parts of geometry are not concerned with distance or measure. This aspect of geometry is called non-metric because of its "no-measure" property. An examination of non-metric properties considers points, lines, planes, geometrical figures, and shapes in space. Such a study enables us to accomplish the following:

1) to introduce geometric ideas and ways of thought;

2) to develop more familiarity with the terminology and notation of "sets" and geometry;

3) to encourage precision of language and thought;

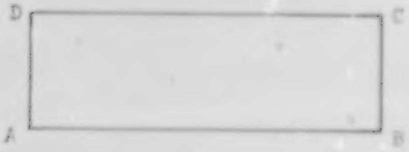
4) to develop spatial perception.

10.1 Sketching

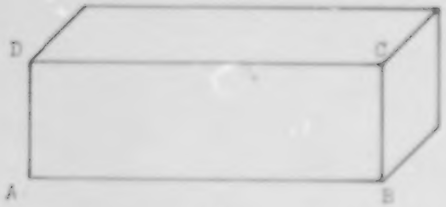
In order to discuss and "draw pictures" of what we will be studying, let us examine a few techniques of representing surfaces and shapes.

Representing points gives us little difficulty and representing lines becomes bothersome only when we try to look at them in perspective. Solid figures, in general, are not difficult to sketch with a little practice.

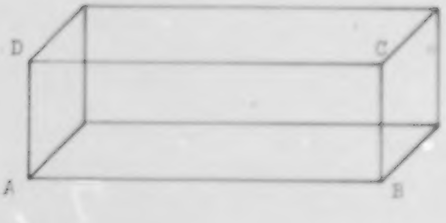
Suppose we start by drawing a box. We may consider the following rectangle ABCD as a representation of a box. This is the view from "head on."



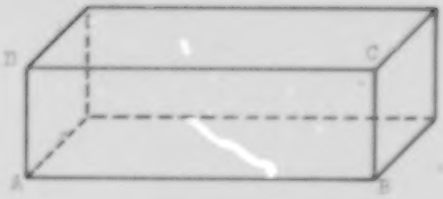
If we think of rotating the box, or equivalently, moving to the right and standing up so that we look down at an angle at one corner of the box, the sketch looks somewhat like this:



Further, if we think of this shape as made of toothpicks, tinkertoys, or rods instead of being solid we would see the "back edges" and the sketch would resemble this:

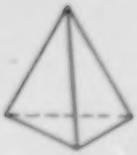


Since we now seem to have created some sort of an optical illusion, where it is not clear which is front or which is back, we make the "back lines" or hidden lines dotted to differentiate them from the others.

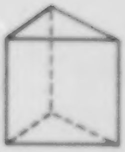


Class Exercises

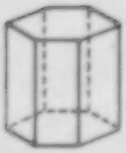
1. Sketch a cube (all faces are squares). Show hidden lines as dotted.
2. Using the figure above:
   (a) Sketch only the top of the box.
   (b) Sketch the bottom and left side.
   (c) Sketch the top and right side.
   (d) Sketch the bottom and both sides.
3. Below are some common solid figures with their names. Sketch them, without tracing, on a larger scale.



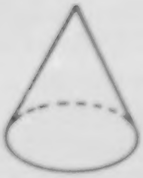
tetrahedron



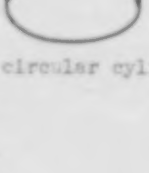
right triangular prism



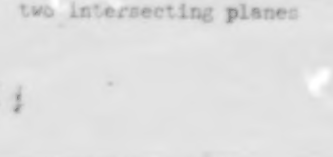
right hexagonal prism



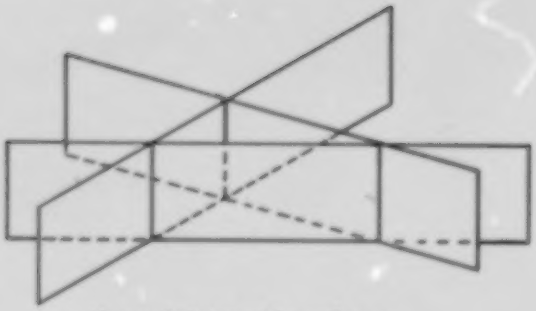
right circular cone



right circular cylinder



two intersecting planes



three intersecting planes

10.2 Points

Let us return to our discussion of points, lines, planes, and space and consider some of their properties and relationships. As mentioned before, we will limit ourselves to those aspects of the problem which do not concern measurement.

What do we mean in mathematics when we use the word point? This is one of the words or terms of mathematics which we use to name an abstract concept or idea. We do not try to define a point but rather discuss its properties and characteristics. We then use this word to define other terms.

Note: The problem of definition in mathematics is not solved by a dictionary approach. When we attempt to define any word, we must use other words. These then also need definition, which requires still more words. In attempting to define all words in this manner we ultimately will have to use a word that we have previously defined. This then gives us what we call a circular definition; i.e., defining word A in terms of word B and word B in terms of word A. Such circular definitions are of no value, for unless we can get outside the circle by somewhere pointing to the actual object, we are unable to do more than use one word for another. Imagine yourself with a French dictionary, no knowledge of the French language, and a French word for which you wish the meaning. Finding this word in the French dictionary only gives you other French words which in turn are defined in French words, and so on.

For this reason, in mathematics we agree to accept some words as "primitive" or "undefined words" and then use these to frame the definitions of other words. Students find it interesting to take a word, find its definition in the dictionary and continue looking up the key words until they find the original word used. Examples are easy to find. In one dictionary *point* is given as a "narrowly localized place having a precisely indicated position." The key word in this definition, *position*, is given as "the *point* or area occupied by a physical object." The same dictionary defines *length* in terms of dimension, *dimension* in terms of extension, *extension* as the act of extending, and extend as "to stretch out to fullest *length*."

Because of this problem of definition we will not attempt to define the terms point, line, plane, or space. We will, however, state formally some axioms, here called properties, which will describe these geometrical objects. Using these "properties" or axioms, it will be possible to learn more about points, lines, and planes. Recall that in Chapter 4 you did not know what many of the elements and operations "really" were, but from their definitions as given in tables much information about their behavior was deduced.

A *point* might be described as a location in space. But this leads us to the circular definition mentioned earlier, for we will use the term point in our definition of space. The idea of a point is suggested by the tip of a sharp pencil, by a dot on a paper or chalkboard, or the period at the end of a sentence. All these are merely representations of points, and not points themselves. The smaller the dot or period, or the sharper the pencil, the better the representation. We usually represent points by dots and label them with capital letters.

10.3 *Sets of Points*

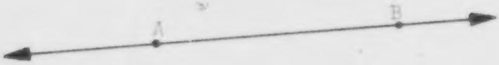
We may think of *space* as the set of all points and examine certain special subsets of space; i.e., sets of points which are the elements of geometry we wish to examine. One of the first of these is a *line*. By line we mean a set of points with certain properties. We will use the word "line" to mean "straight line." Just as a point was represented by the tip of a pencil, a line is represented by the edge of a ruler, a string stretched taut between two points, or the "line of sight" of the surveyor.

Although at times we refer to a portion of a line, we must be careful to make it clear whether we mean the entire line or not. Later we will introduce some notation to help clarify this situation. Again, as with points, our marks on paper, chalkboard, and the like, will be only representations of lines. We will often label lines with lower case script letters as line ℓ, m, or n.

One of the simplest and most basic properties of space is represented by the uniqueness of a line drawn through two points on the chalkboard, or the fact that two pieces of string stretched between two points follow the same path (as far as physically possible). Unique, as used here, means "exactly one."

Property 1: Through any two different points in
space there is exactly one line.

Another method of labeling or naming lines is dependent upon this property. If A and B are any two distinct points both on a line, or if a line passes through the points A and B, then we use the symbol $\overleftrightarrow{AB}$ to denote such a line.



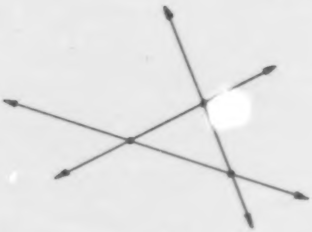
If three or more points are contained in the same line, then we say such points are collinear. Thus, points A, B, and C on the line below are collinear.

When more than two points on a line are named, we have many ways for naming the line. We might name the line above as $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BA}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{CB}$.

Class Exercises

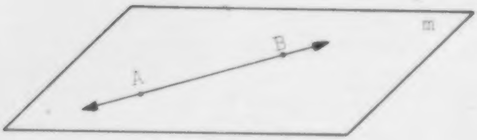
4. With two points only one line is "determined," while three non-collinear points determine three lines. Four points, no three of which are collinear, determine how many lines? Five points? Can you discover a formula which will give the number of lines determined by n points, no three of which are collinear? Complete the following table:

Number of Points	Number of Lines
2	1
3	3
4	
5	
6	
n	



Three points-three lines

Another basic concept of geometry is that of a plane. Like the line, this is also a set of points in space with certain properties. Intuitively, we think of a plane as having the property we have in mind when we use terms like flat, level, even, and smooth. We will attempt to make this "flatness" more precise a little later. We think of the surface of the chalkboard or our paper as representing a plane surface. If we wish to represent a plane in perspective with a sketch, we draw only a portion, as with a line. We indicate a plane by a figure like that below and label it with a lower case letter as shown. Remember, although the picture appears to be bounded, the plane it represents continues on indefinitely in the directions indicated.

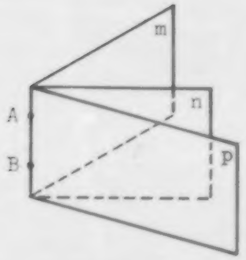


The flatness of the plane and the straightness of a line suggest that if two points, A and B, of a line are in plane m, then every point of the line through A and B lies in the plane. We may state this formally as a second basic property of space as follows:

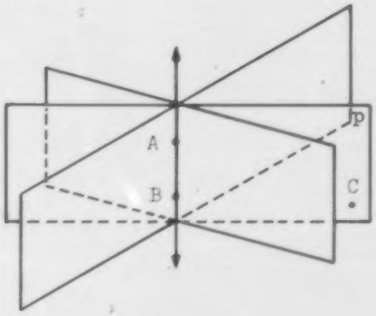
> Property 2: If a line contains two different points of a plane, then the line lies in the plane.

This property may be stated a variety of ways. We may say that the plane contains the line, or that every point of the line is a point of the plane, or that the line is a subset of the plane.

Recalling Property 1, that exactly one line is determined by two points, we may wonder about points and planes. Given two points, how many planes are determined? Since these two points will determine a unique line we are also asking how many planes contain a single line. If we think of the hinges of a door as the two points, and the different positions of the door as representing different planes, we see that any number of planes may pass through two points or equivalently through one line.



Just as there is only one position for the door when it is closed, i.e., the two hinges and the latch determine one position, three points will determine one plane. Considering the same two points A and B with a third point C not on $\overleftrightarrow{AB}$, then only one plane, labeled p in the figure below, contains all three points.



This is another of our basic properties and we state it formally.

> Property 3: Any three points not in the same straight line are in exactly one plane.

We see from this property that three points not in the same straight line can be used to name a plane since they locate exactly one plane. In the figure plane p may be named plane ABC.

This property also explains the reason for the stability of such things as tripods and three-legged stools. You may illustrate this by demonstrating the ease of supporting a book (plane) with three fingers (points) as contrasted with two fingers (points).

Class Exercises

5. How many planes are determined by the ends of the four legs of a table? Does this help explain why the legs of a table must be the correct length in order to sit steady whereas a tripod always sits steady? Must the legs of a table always be the same length?

10.4 Intersections of Lines and Planes

Since the elements of geometry are sets of points, we have all the previously defined properties of sets at our disposal. In the chapter on sets, the term intersection was defined precisely, and we agreed that whenever we used this word it would have exactly the meaning given in the definition. This is what we do with all technical words in mathematics. Once they have been defined they will always have the same meaning and be used in the same way. Sometimes, however, a technical word in mathematics, carefully defined in one way, may also be an everyday word used in a somewhat different sense. Such a word is intersection. When used with sets, intersection means only one thing, the set of all elements common to two sets. This is the meaning given earlier and will continue to be the meaning of intersection of sets. From this definition we also developed the empty set, $\emptyset$, which we defined to be the set with no elements. Thus, the intersection of two disjoint sets is the empty set.

In everyday usage we often speak of a "street intersection" or "two paths intersecting." This meaning is similar to the "points in common" definition given above, but in everyday usage if two streets do not meet we say they do not intersect, rather than say "their intersection is the empty set."

Geometry and the language of sets developed as two different disciplines at two different times. This helps explain the use of the same word in two different ways. If we keep these two uses of the word in mind, then a statement like the following is meaningful:

> If two lines do not intersect, their intersection is the empty set.

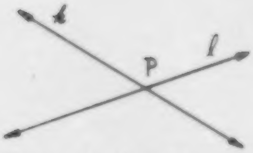
Although we will try to avoid statements of this type, and the meaning will usually be clear from the usage, the teacher should be aware of the difference and alert to the possibilities of confusion on the part of the student.

Two Lines

What possibilities exist for two lines in space? If they intersect, (by this we mean the intersection is not the empty set), they have at least one point in common. What if they have two points in common? Then by Property 1, they must have all points in common, or we say they are coincident.

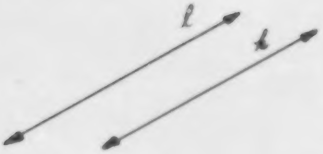
Note: Two lines whose intersection is not the empty set lie in the same plane. Why? The possible arrangements of two different lines may be described in three cases.

Case 1. ℓ and k intersect, or $\ell \cap k$ is the point P and not the empty set.



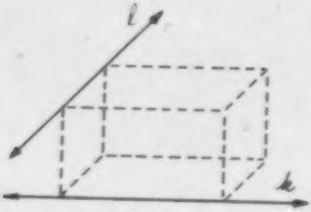
intersecting lines

Case 2. ℓ and k do not intersect and are in the same plane. $(\ell \cap k = \emptyset)$ Such lines are said to be parallel lines.



parallel lines

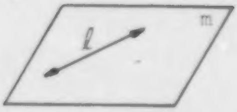
Case 3. ℓ and k do not intersect and are not in the same plane. $(\ell \cap k = \emptyset)$ Such lines are said to be skew lines.



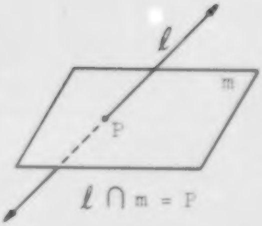
skew lines

A Line and a Plane

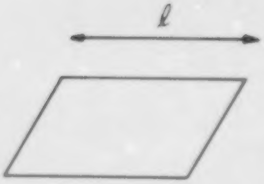
A little thought will reveal the three possible arrangements that may exist for a line and a plane in space. Property 2 tells us that if two or more points of a line are contained in a plane, then all points are so contained, and the line lies completely in the plane. Nothing, however, prevents a line and a plane from having one or no points in common. In the former we say the line intersects the plane and in the latter we say the line is parallel to the plane.



$\ell \cap m = \ell$



$\ell \cap m = P$

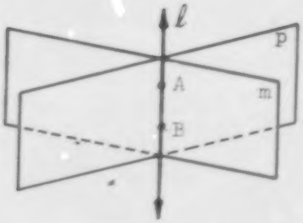


$\ell \cap m = \emptyset$

Two Planes

If we consider two planes, one possible relationship is that of coincidence. Let us confine our attention to two different planes, i.e., not coincident, and ask what possibilities exist. Either their intersection is empty, so that we say the planes are parallel, or the intersection is non-empty.

In the latter case our intuition and previous efforts at sketching probably led us to expect a line as the intersection. Can we make this conclusion more plausible by using our previously developed properties?



If A and B are distinct points, both contained in the intersection of m and p, then by Property 1 they are contained in exactly one line, say ℓ. But since A and B are both in plane m, Property 2 tells us that line ℓ is in m.

The same reasoning puts ℓ in plane p. All this seems to add weight to our conjecture that the intersection is a line. Note, however, that we have assumed the distinctness of the two points A and B. We have not really proved that the above conclusion is true but let us accept it as another basic property of space, just as we did the previous three.

> Property 4: If the intersection of two different planes is not empty, then the intersection is a line.

This property forces the mathematical concept of a plane to agree with our intuitive concept of plane. Without Property 4, we have no mathematical reason to rule out the possibility of two planes intersecting in a single point. This, of course, contradicts our intuitive notion of two planes intersecting.

Class Exercises

6. How many examples of intersections of planes to give straight lines can you see in your immediate surroundings? The intersection of two walls? A ceiling and a wall? The edges of a desk?

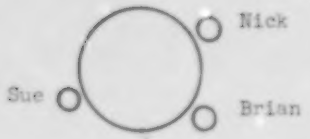
7. Find some examples around you of intersecting lines and planes.

8. Consider the line determined by a point on the light switch and a point on the pencil sharpener. Does this line intersect anything inside the room? Outside the room?

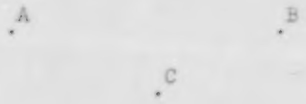
9. Consider the plane determined by a point on the light switch, a point on the pencil sharpener, and some third point in the room. Is a single plane determined? Where does this plane intersect the walls of the room? Where does it intersect the ceiling? Does it intersect the instructor? Is anyone decapitated?

10.5 Segments and Unions of Sets

We use the word "between" in referring to points located in certain ways. How are points arranged when we say that one point is between two others? With people seated around a table it is difficult to say who is between whom. Is Nick between Sue and Brian, or is Sue between Brian and Nick, or both?

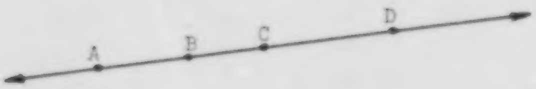


What about three points arranged as shown?



Can we agree that any one of the points is between the other two? In a situation of this nature "between" does not seem to apply.

If the points in question are on the same line as A, B, C, and D,



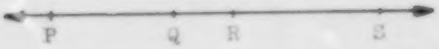
then we have no difficulty in our use of the word between. We say that B is between A and C, (or A and D), C is between B and D (or A and D) and both B and C are between A and D. Thus, when we say one point is between two others, we are implying that the points are collinear.

We may use the above idea when we wish to speak of a portion of a line. We call the set of points consisting of A and B and all points between A and B, segment AB and write it as $\overline{AB}$. Note the difference in notation between $\overline{AB}$ (segment) and $\overleftrightarrow{AB}$ (line).

Another item we wish to recall from previous work is the union of two sets. Remember that this is the set of all elements belonging to at least one of the two original sets. In the figure above the union of $\overline{AB}$ and $\overline{BC}$ is $\overline{AC}$. This concept is used in the following class exercises.

Class Exercises

10. Examine line $\overleftrightarrow{PS}$.

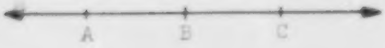


Name two segments:

(a) whose intersection is a point.
(b) whose intersection is a segment.
(c) whose union is a segment.
(d) whose union is two segments.
(e) whose intersection is empty.

11. How many segments are in the figure in Exercise 10?

12. Simplify the following by referring to line $\overleftrightarrow{AC}$.

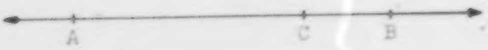


(a) $\overline{AB} \cap \overline{BC}$
(b) $\overline{AC} \cap \overline{BC}$
(c) $\overline{AB} \cup \overline{BC}$
(d) $\overline{AB} \cup \overline{AC}$
(e) $A \cap \overline{BC}$
(f) $A \cup \overline{AB}$
(g) $\overleftrightarrow{AC} \cap (\overline{AC} \cap \overline{BC})$
(h) $B \cap \overline{AC}$

13. Draw two segments $\overline{AB}$ and $\overline{CD}$ so that $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is not empty.

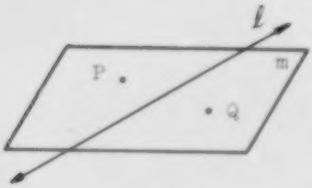
10.6 Separations

A point on a line separates the line into two parts. Each part is called a *half-line*. Thus, AB is separated into two *half-lines* by the point C in the following diagram. Notice we have *three* subsets of the line, the two half-lines and the point of separation.



We speak of the half-line containing A or the half-line containing B. A half-line together with its end-point is called a *ray*. Thus, the union of point C with the half-line containing point B is a ray, written $\overrightarrow{CB}$. Note the notation used and contrast it with the notation used for line and line segment. In the latter two cases order made no difference. Thus, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ both denote the same line, $\overline{CD}$ and $\overline{DC}$ name the same segment. Order, however, is important when considering rays. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ do not mean the same ray. The first letter names the end-point while the second letter names some other point on the ray. Ray $\overrightarrow{AB}$ starts at A and contains B; ray $\overrightarrow{BA}$ starts at B and contains A.

A similar situation holds with a line in a plane. The line separates the plane into two half-planes. In the following figure, line ℓ separates plane m into the two half-planes containing P and Q, respectively.



Line ℓ belongs to neither half-plane, but forms the boundary of each. Note that the line divides the plane into three subsets, the two half-planes and the line itself.

Space is separated into two half-spaces by a plane. Here again we say that the plane belongs to neither half-space.

Class Exercises

14. Draw a line containing the three points P, Q, and R, with R between P and Q. Use the diagram to simplify the following.

(a) $\overrightarrow{PQ} \cap \overrightarrow{QP}$
(b) $\overline{PR} \cup \overrightarrow{RQ}$
(c) $\overline{PR} \cup \overline{RQ}$
(d) $R \cap \overline{PQ}$
(e) $P \cap \overrightarrow{RQ}$

15. If points A and B are in the same half-space formed by plane m in space, what possibilities exist for $\overleftrightarrow{AB} \cap m$?

10.7 Conclusion

What major ideas have we covered in this chapter? We have looked at geometric elements as ideas and seen that we do not put points, lines, and planes on the board but only representations of such ideas. We have seen that only some elements of geometry are defined, whereas some are left undefined. These we use as our building blocks to develop more complex ideas.

We have seen how points, lines, and planes in space are related. We have discussed the intersections and unions of these various geometrical elements.

In the next chapter we will continue this approach and use these basic elements of point, line, and plane to develop other geometrical figures.

Chapter Exercises

1. Sketch two planes, m and n, that intersect in line ℓ.

2. Given two sets, one with eight elements and one with twelve elements,
   (a) what is the maximum number of elements in their intersection? The minimum?
   (b) What is the maximum number of elements in their union? The minimum?

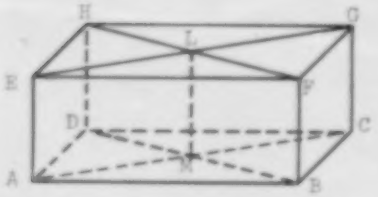
3. If m and ℓ denote a plane and a line, respectively, draw a sketch to show each of the following situations:
   (a) $m \cap \ell = \emptyset$
   (b) $m \cap \ell = \ell$
   (c) $m \cap \ell =$ point A

4. If m, n, and p denote planes, draw a sketch to show each of the following:
   (a) $m \cap n =$ line ℓ
   (b) $m \cap n = \emptyset$
   (c) $m \cap n \cap p =$ line ℓ
   (d) $m \cap n \cap p =$ point A

5. How do a ray and a half-line differ?

6. How do $\overleftrightarrow{AB}$, $\overline{AB}$, $\overrightarrow{BA}$ and $\overrightarrow{AB}$ differ?

7. Consider the accompanying sketch, and the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points.

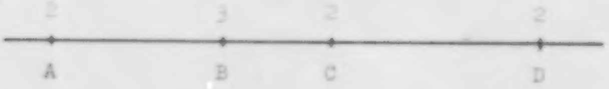


Name the following:

(a) A pair of intersecting planes
(b) A pair of parallel planes
(c) Three planes that intersect in a point
(d) Three planes that intersect in a line
(e) A pair of parallel lines
(f) A pair of skew lines
(g) Three lines in the same plane that intersect in a point
(h) Three lines not in the same plane, that intersect in a point
(i) Four planes that have exactly one point in common.

8. How many planes are determined by a line and a point? Must any conditions be placed on the line and point for the answer to be unique?

9. Four houses, A, B, C, D are on the same street with two boys living in house A, three in B, two in C, and two in D, as shown below.

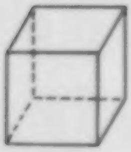


If the boys form a club, at which house should meetings be held in order to minimize walking?

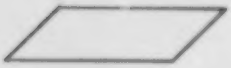
Answers to Class Exercises

Other orientations are possible.

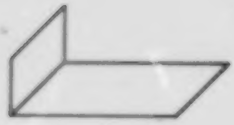
1.



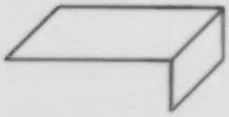
2. a)



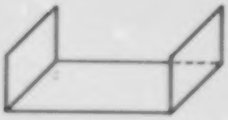
b)



c)



d)



4.

Number of Points	Number of Lines
2	1
3	3
4	6
5	10
6	15
N	$\frac{1}{2}N(N - 1)$

5. Since the ends of any three of the table legs determine a plane, a total of four planes are possible. The three points of the tripod determine only one plane. The ends of the table legs need only lie in the same plane, and thus not necessarily be the same length.

6.
7.
8.
9.

Answers will depend upon the situations. These questions are designed to help you visualize lines and planes in space.

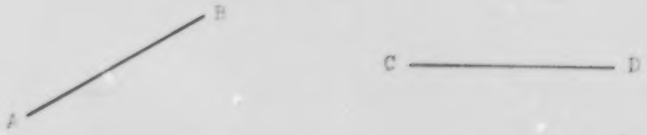
10. Several answers are possible.

(a) $\overline{PQ}$ and $\overline{QR}$
(b) $\overline{PR}$ and $\overline{QS}$
(c) $\overline{PQ}$ and $\overline{QR}$
(d) $\overline{PQ}$ and $\overline{RS}$
(e) $\overline{PQ}$ and $\overline{RS}$

11. 6

12. (a) B
(b) $\overline{\text{[illegible]}C}$
(c) $\overline{AC}$
(d) $\overline{AC}$
(e) ∅
(f) $\overline{AB}$
(g) $\overline{BC}$
(h) B

13.



14. (a) $\overline{PQ}$
(b) $\overrightarrow{PQ}$
(c) $\overline{PQ}$
(d) R
(e) ∅

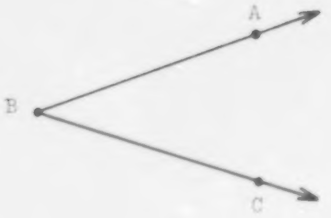
15. Either one point, or the empty set.

Chapter 11

NON-METRIC GEOMETRY, II

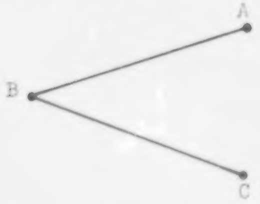
11.1 Angles and Triangles

You are familiar with the terms angle and triangle. How do we define and use these words in geometry? We define *angle* as the union of two rays with the same end point, not both on the same line. The common end point is called the *vertex* of the angle and the rays are called the *sides* of the angle. Thus, in the figure below, angle ABC, written $\angle ABC$, is composed of the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ with point B in common.



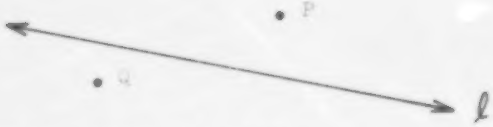
$$\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$$

In the symbol "$\angle ABC$" the letter in the middle always names the vertex. $\angle ABC$ and $\angle CBA$, however, both indicate the same angle. Notice that the angle is composed of rays, not segments. A figure such as the one shown below is not, by our definition, an angle.



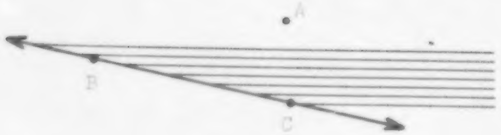
The figure does, of course, determine an angle in that segment BA suggests ray BA and segment BC suggests ray BC. These rays, then, give us an angle as defined.

An intuitively simple aspect of an angle is the "inside" or "interior" of the angle. Probably every student could point to the area or region we have in mind when we use such a word. An angle divides a plane into two regions and in some sense of the word we mean the smaller of the two regions. Describing such an area in terms of our previous ideas involves the careful use of language, if we are to say exactly what we mean and nothing else. Recall that a line separates a plane into two regions. Thus, given a situation like the one shown,

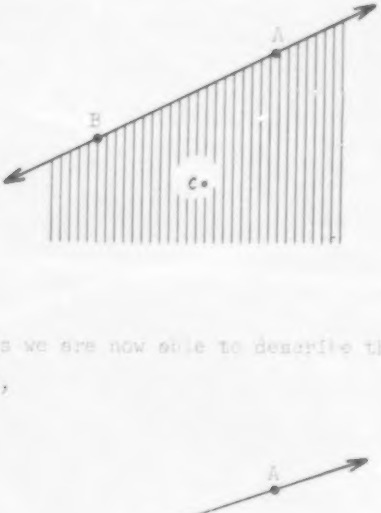


we may use points of the plane to identify the two regions, that is, the two half-planes. Let P and Q be points such that the intersection of the line ℓ and $\overline{PQ}$ is between P and Q. Then P and Q are on opposite sides of line ℓ. By the term "P-side of line ℓ", we mean the half-plane that contains the point P.

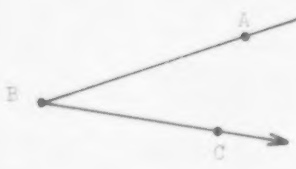
In the following figure the horizontal lines indicate the A-side of $\overleftrightarrow{BC}$.



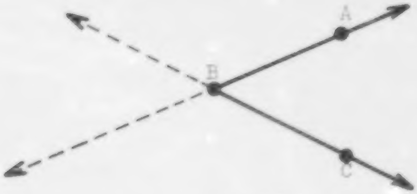
In a similar manner, the vertical lines in the following figure indicate the C-side of $\overleftrightarrow{AB}$.



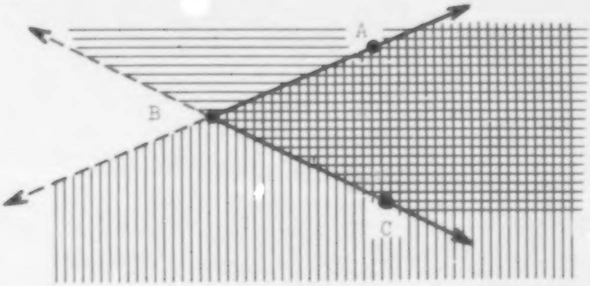
With these ideas we are now able to describe the interior of an angle. Given the angle ABC,



the rays BA and BC determine the lines BA and BC as shown.

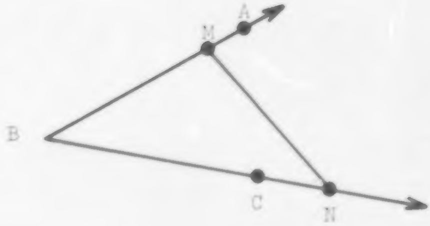


If we again refer to the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{BA}$, then the intersection of these two regions, doubly shaded, is what we mean by the interior of $\angle ABC$.



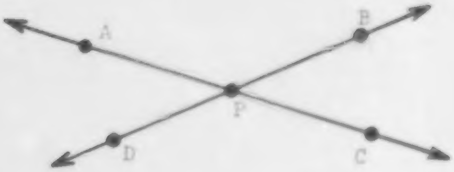
Formally, we say the interior of $\angle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$.

Still another way of defining the interior of an angle is to take any point, M, on $\overrightarrow{BA}$ and any point, N, on $\overrightarrow{BC}$. These two points determine a segment, $\overline{MN}$, as shown.



We may define the interior of $\angle ABC$ to be the union of all such segments with the exception of their endpoints. Why are these definitions of the interior of an angle not the same if we include the endpoints of the segments?

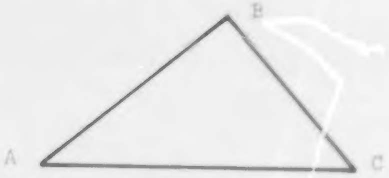
When we consider two intersecting lines,



we see that the resulting rays form four angles. We call a pair of opposite angles, such as $\angle$ BPC and $\angle$ APD, *vertical angles*. Notice that their sides form two pairs of opposite rays. The figure also contains another pair of vertical angles, $\angle$ APB and $\angle$ CPD.

Triangles

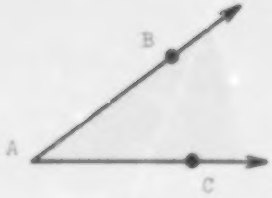
Three non-collinear points, A, B, and C, will determine three segments, $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. The union of these three segments is called *triangle ABC* and is written "$\triangle$ ABC". The segments are called *sides* of the triangle. The points A, B, and C are called *vertices* (plural of vertex) and angles $\angle$ ABC, $\angle$ BAC and $\angle$ ACB determined by triangle ABC are called the *angles of the triangle*.

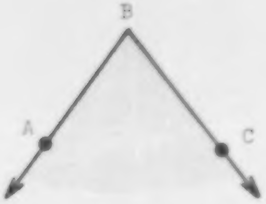


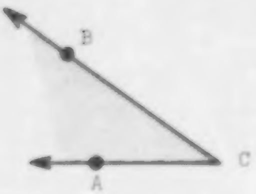
$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}$

Note carefully the definition. It is the union of segments, not lines or rays. Although the segments $\overline{AB}$ and $\overline{AC}$ determine the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and thus determine the angle BAC, the segments themselves do not form the angle. This is why we say that a triangle determines or locates three angles, but that the angles are not themselves part of the triangle.

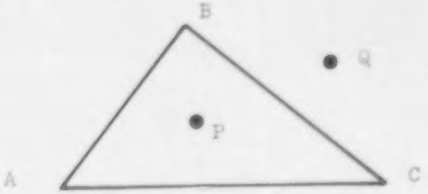
A triangle also separates a plane into two regions which we call the interior and exterior of the triangle. Here again we have three subsets of the plane, the triangle, its interior, and its exterior. We may use the interior of the angles of a triangle to define the interior of the triangle. The three angles determined by $\triangle$ ABC each have interiors as shown.





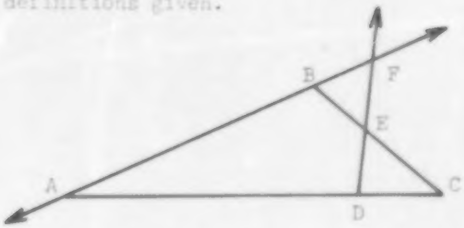


By using the intersection of these three sets, we may define the interior of a triangle as the intersection of the interiors of the three angles of the triangle.



This definition puts the point P in the interior of the triangle shown above, since it is in the interior of each of the angles. Point Q is not in the interior of the triangle, although it is in the interior of $\angle$ BAC. If a point is in the interior of two angles of a triangle is it in the interior of the third angle?

A diagram like the one below may help students understand the meaning of the different definitions given.



You may ask students to shade regions such as interior $\triangle$ ABC $\cap$ interior $\triangle$ ADF, or interior $\triangle$ ABC $\cup$ interior $\triangle$ ADF. Or you may ask them to identify the points in the union and intersection of sets of points as follows:

a. $\overleftrightarrow{AB} \cap \triangle ABC$ ($\overline{AB}$)
b. $\angle ACB \cap \overline{BA}$ (points A and B)
c. $\overrightarrow{BA} \cap \overline{BC}$ (point B)
d. $\overrightarrow{BA} \cup \overline{BF}$ ($\overrightarrow{FA}$)
e. $\triangle ABC \cap$ interior $\triangle ABC$ ($\emptyset$)

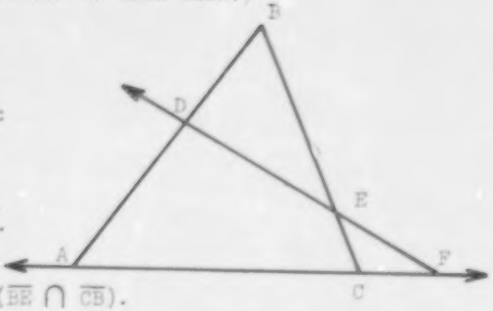
Class Exercises

1. Define the exterior of an angle and of a triangle. (Make use of the fact that the interior has been defined in each case.)

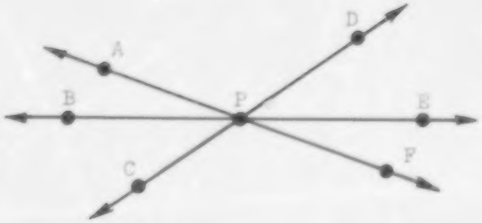
2. Refer to the figure below.

   Describe the set of points in:

   a. $\triangle DBE \cap \triangle BCF$.
   b. $\angle BAC \cap \overline{BC}$.
   c. the interior of $\triangle BDE \cap \overrightarrow{FD}$.
   d. $\overline{AB} \cap \overline{BC}$.
   e. $(\overline{AB} \cap \overline{BD}) \cup (\overline{DE} \cap \overline{FD}) \cup (\overline{BE} \cap \overline{CB})$.
   f. $\angle BAC \cap \angle BCA$.



(Note: Exercises 3-5 refer to the following figure.)



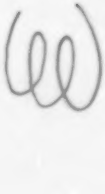
3. Name four pairs of vertical angles. Are there others?

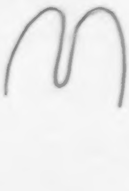
4. Name three half-planes.

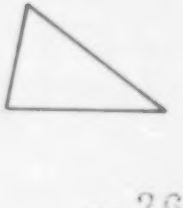
5. What is $\angle DPE \cap \angle EPF$?

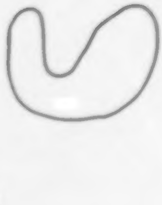
11.2 Simple Closed Curves

The word "curve" is another word which we use in both everyday language and in our mathematical language. Like many other words, the two usages do not agree in all respects. Below are some representations of curves.



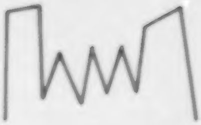


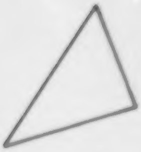


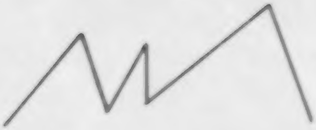


A plane curve is a set of points, all in one plane, which can be represented by a pencil drawing made without lifting the pencil from the paper. Segments and triangles are both examples of plane curves. Note that a straight line is also a curve. It is this technical usage that does not agree with our general usage where curve is associated with the concept of changing direction.

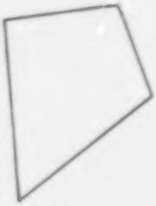
Curves made up of line segments are called broken-line curves.



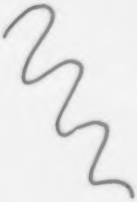




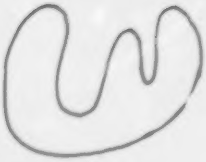
These are often encountered in the graphical representation of data where they are called broken-line graphs. A curve which can be represented by a figure which starts and stops at the same point is a closed curve. Furthermore, if the curve passes through no point twice, then it is called a simple closed curve. Notice that a simple closed curve does not necessarily have a "nice" shape, but only that it is closed and does not cross itself. In the examples below, all are curves; (a), (c), and (d) are closed curves; (a) and (c) are simple closed curves.



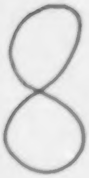
(a)



(b)

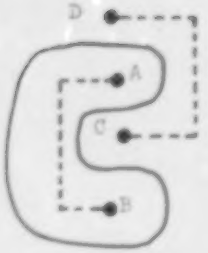


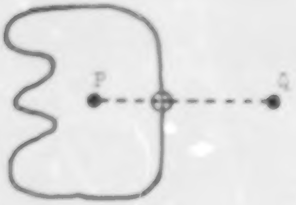
(c)



(d)

A property of simple closed curves, that seems intuitively obvious, is that such a curve separates the plane into exactly two regions giving three subsets of the plane. Any two points in the interior, such as A and B in the figure below, may be joined by a broken-line curve that does not cross the simple closed curve. A similar statement holds for the exterior and the points C and D. Also, the segment connecting any point of the interior, P, with any point of the exterior, Q, must intersect the curve at least once. This information is contained in the Jordan Curve Theorem which states that a simple closed curve separates the plane into exactly two regions.

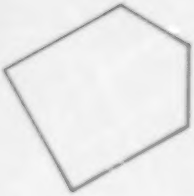




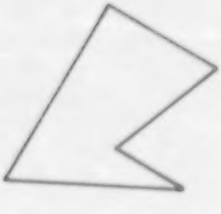
We refer to the curve itself as the boundary of the interior, or the boundary of the exterior. The interior is also called a *region*; the union of the interior with its boundary is called a *closed region*.

We may use the concept of simple closed curve to restate the definition of a triangle more concisely. "A triangle is a simple closed curve which is the union of three segments."

There are, of course, many kinds of simple closed curves. One important group of these, which includes the triangle, is the set of polygons. A *polygon* is a simple closed plane curve composed of the union of line segments. As with triangles, we refer to the segments of polygons as *sides*; the angles determined by the sides are called the *angles of the polygon*; the vertices of the angles are called the *vertices of the polygon*. Polygons are either convex or concave.



convex polygon



concave polygon

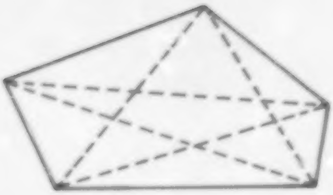
A polygon is said to be *convex* if each of its sides lies in the boundary of a halfplane which contains the rest of the polygon. If we think of extending any one side then the remainder of the polygon will be contained in only one of the resulting half planes.

Polygons are classified in several ways: one of the simplest is by the number of sides. Polygons with four sides are called *quadrilaterals*; polygons with fives sides are called *pentagons*. A few polygons with their names are listed below:

Name	Number of sides
Triangle	3
Quadrilateral	4
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10

Other polygons have names, but such names are seldom used. A project many students find interesting is to discover names for as many polygons as possible, explaining the word stems.

A segment connecting any two non-adjacent vertices is called a diagonal of the polygon. Triangles have no diagonals while quadrilaterals have two. From the sketch below we see that a pentagon has five diagonals.

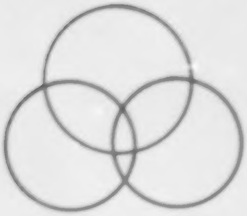


Class Exercises

6. Complete the following table and find a formula for the number of diagonals in a polygon of n sides, $n \geq 3$.

Number of sides	Number of diagonals
3	0
4	2
5	5
6	
7	
8	
.	
.	
.	
n	

7. How many simple closed curves can you find in the figure below?



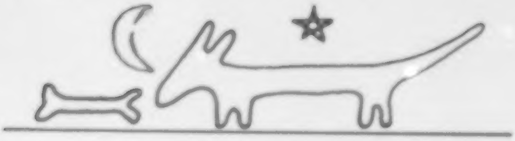
8. What is wrong with using the term "curved line"?

9. Identify each of the figures below as one of the following:

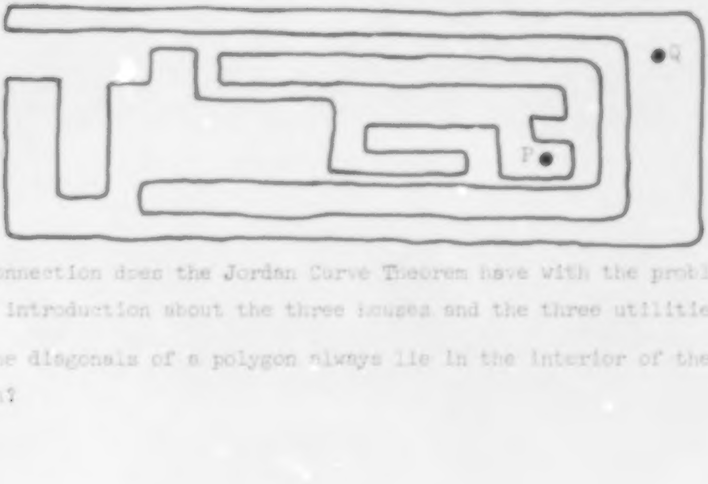
   a. closed curve, not simple

   b. curve, not closed

   c. simple closed curve



10. Are P and Q in the interior or exterior of the curve below?

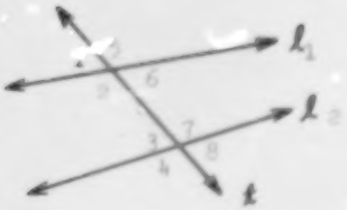


11. What connection does the Jordan Curve Theorem have with the problem in the introduction about the three houses and the three utilities?

12. Must the diagonals of a polygon always lie in the interior of the polygon?

11.3 Transversals, Parallels, and Parallelograms

When two lines in a plane are both intersected by a third line, then the third line is called a transversal. Such a situation is shown below where line t is the transversal.

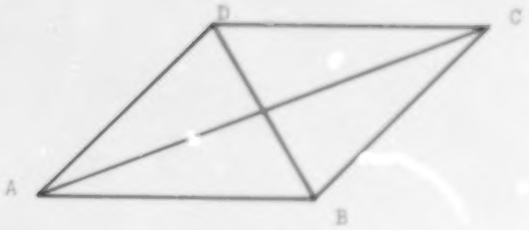


Here the angles have been identified by the use of numerals written in the interior of the angles. This is a use of numerals that we have not encountered before. They are being used as labels or names, much as a Social Security number, room number, or a telephone number can be used as a name.

Many of the pairs of angles formed by a transversal are encountered so often that we give them special names. Pairs of angles such as 1 and 3, are called corresponding angles. Angles 6 and 8 are also corresponding angles. Do you see two other pairs of corresponding angles?

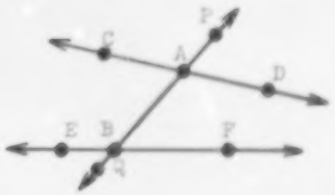
Angles such as 3 and 6 are called alternate-interior angles. Can you see any rationale behind such a name?

When two pairs of parallel lines intersect, the figure formed by the resulting segments is called a parallelogram. A parallelogram is also defined as a quadrilateral whose opposite sides lie on parallel lines. (Here opposite means non-intersecting.) We write ▱ ABCD for parallelogram ABCD.



In the figure above segments $\overline{BD}$ and $\overline{AC}$ are diagonals of ▱ ABCD. Parallelograms and their properties will be considered again in Chapter 12.

Class Exercises



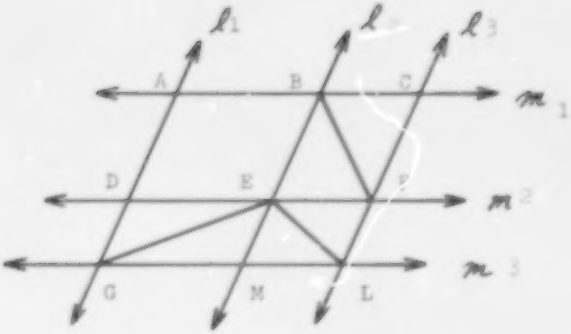
13. Using the figure above, name:

   a. four pairs of corresponding angles

   b. two pairs of alternate-interior angles

   c. four pairs of vertical angles

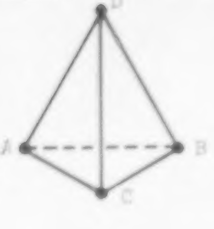
14. If ℓ_1, ℓ_2, and ℓ_3 are parallel and m_1, m_2, and m_3 are parallel, find a parallelogram which is partially in the interior of another parallelogram. How many parallelograms can you see in the figure? How many diagonals? Triangles?



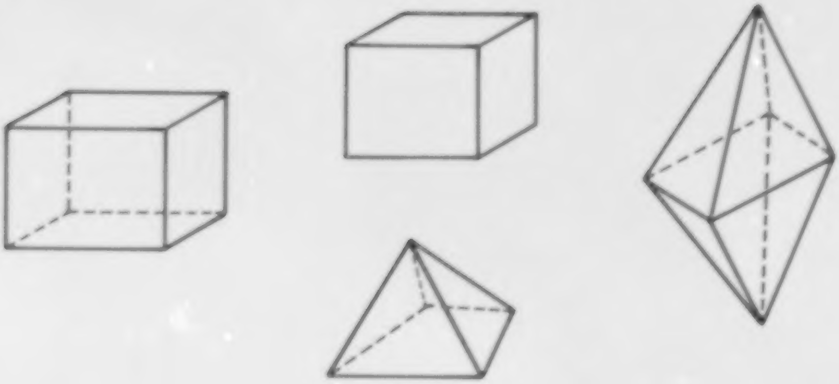
11.4 Solids

We have examined many subsets of the plane, such as lines, angles, triangles, and polygons. There are various other subsets of space, not subsets of a plane, that we will consider. If we use our lines and planes as building blocks, a variety of solids may be formed.

By Property 3 of the last chapter, we know that any three non-collinear points determine a unique plane. A fourth point not in the plane of the first three will determine a space figure called the *tetrahedron*. We may define a tetrahedron as the union of the four triangular regions determined by four points in space, not in the same plane. In tetrahedron ABCD below, the four points A, B, C, and D are called the *vertices*, the segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ are called *edges*, and the four triangular regions formed are called *faces*.



The tetrahedron is an example of a class of three dimensional objects known as polyhedrons. Other representations of polyhedrons are shown below.

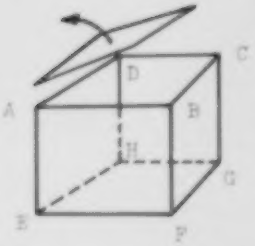


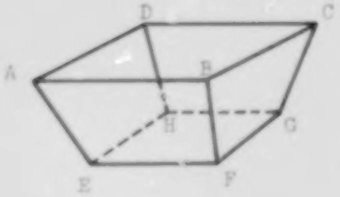
Just as polygons separate the plane, polyhedrons separate space. Space is divided into three subsets, the interior of the polyhedron, the exterior, and the polyhedron itself.

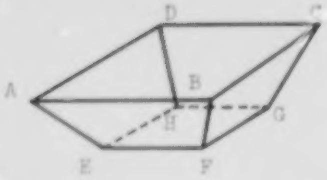
Polyhedrons of the type shown, are called simple polyhedrons and have an interesting relationship among the vertices, edges, and faces. If you will count them in each of the preceding figures, you will find that the sum of the number of vertices and faces is two more than the number of edges. This relationship, $V - E + F = 2$, is known as Euler's formula. This fact is very surprising, and students find it interesting to verify with various solids.

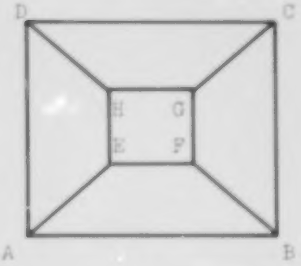
An intuitive proof of Euler's formula may proceed along the following lines. Consider a polyhedron and remove one face leaving the edges and vertices unchanged. Thus, if originally $V - E + F$ is a constant, say $\underline{n}$, then removing one face gives $V - E + F = k$, where $k = n - 1$, and now we wish to find $\underline{k}$. If we think of the polyhedron made of rubber or some very deformable plastic, we may open it out about the missing face so that a plane surface made of polygons results. Although these polygons may be shaped differently than the faces of the original polyhedron, they will be the same in number, and they will have the same number of vertices, edges, and faces. Thus, the numerical value of $V - E + F$ remains unchanged. The argument in the following paragraphs is applied to the cube as a specific example. Notice however that at no step does the argument depend upon any special properties of the cube but applies to simple polyhedra in general.

Removing a face and "opening out" the cube results in the following transformation. We in effect remove the "top" and "flatten out" the remainder to give the plane figure ABCD shown. Notice that although the shapes change, the number of vertices, edges, and faces remains the same. The value of $V - E + F$ is still $\underline{k}$.

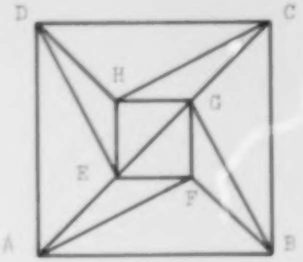




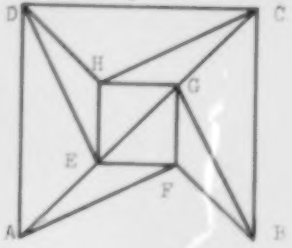




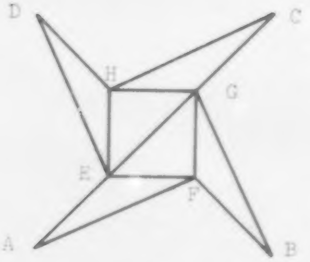
If we draw diagonals in each polygon to subdivide the polygon into triangles, the value of $V - E + F$ remains unchanged, for each diagonal adds one edge and one face. Adding 1 to each of E and F does not affect the total $V - E + F$.



Any triangle like $\triangle ABF$ which has only one side exposed to the "outside" may be removed by deleting side $\overline{AB}$.

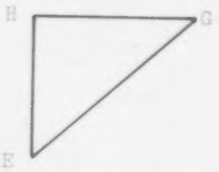


This type of deletion does not change V - E + F for we have decreased both E and F by 1. We may also remove $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$ to give the following:



Deleting a triangle such as $\triangle AEF$, by removing $\overline{AE}$ and $\overline{AF}$, also leaves V - E + F unchanged for this decreases V by 1, E by 2, and F by 1.

One or the other of these two methods of deleting triangles may be carried out until only a single triangle remains. The value, k, of V - E + F has still not changed and at this point we resort to counting.



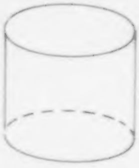
Here we see that V = 3, E = 3 and F = 1 so that V - E + F = k = 1, and since k = n - 1, n = 2. Again recall that the same result would occur had we started with any simple polyhedron other than the cube.

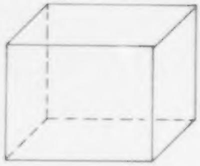
Sometimes this formula is referred to as Descartes' formula since he seems to have preceded Euler in discovering it. The number of vertices and faces of any simple polyhedron is two more than the number of edges.

$$V - E + F = 2$$

Other Solids

You are familiar with cylinders and prisms like the ones shown below.

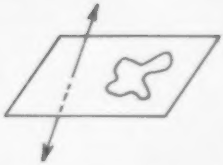




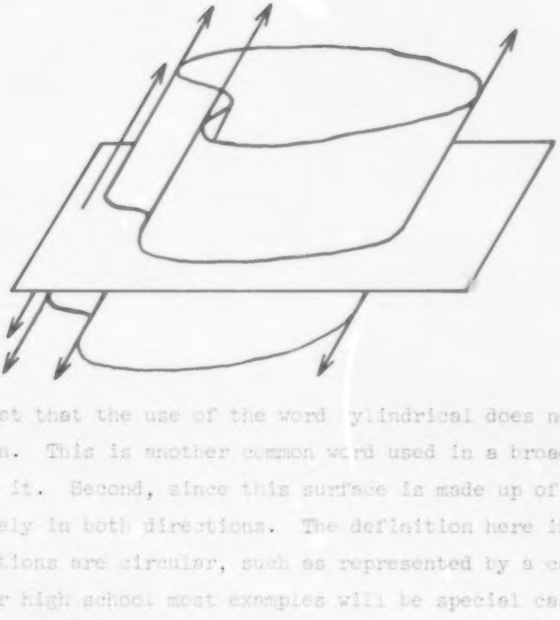
These are examples of special kinds of geometrical solids to be considered in more detail by students who continue to study mathematics. Instead of treating each individual solid as unique, mathematicians use general and broad definitions to encompass whole categories. Here we will indicate a more general development of cylinders and prisms.

Let us examine how such solids may be formed.

Consider any simple closed curve in a plane and a line not parallel to the plane.

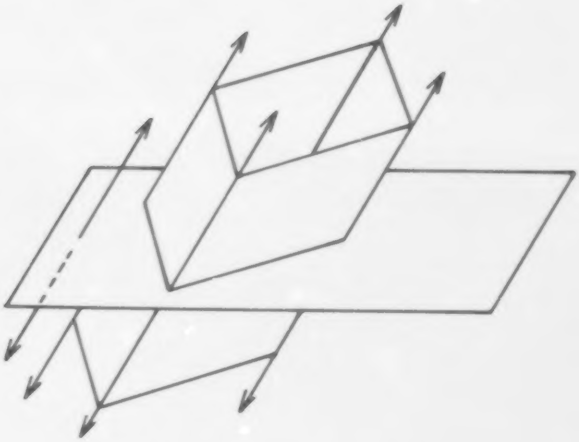


All lines parallel to the original line and passing through the curve will form what is called a *cylindrical surface*.



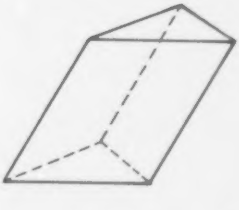
Notice first that the use of the word cylindrical does not imply circular in cross-section. This is another common word used in a broader sense than we normally use it. Second, since this surface is made up of lines, it extends indefinitely in both directions. The definition here includes the case where cross-sections are circular, such as represented by a cardboard mailing tube. In junior high school most examples will be special cases of the more general definition given above. Future work however makes it convenient to have a general definition of this nature.

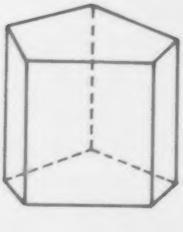
The simple closed curve that gives the surface its shape could be a polygon, but the surface is still called a cylindrical surface.



When two parallel planes intersect such a surface, that portion of the surface between the planes, together with the regions cut from the planes, forms a prism. If the cross-section is not a polygon but some other closed curve, we get what is commonly called a cylinder. (Other definitions of prisms and cylinders that use the concept of congruence are sometimes given in geometry books.)

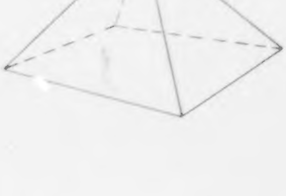
The polygons in the two parallel planes are called bases. Prisms are often classified by their bases. Thus, we have triangular prisms, hexagonal prisms, and so on.

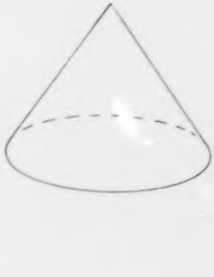




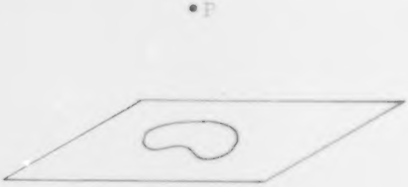
Other classifications are possible and some will be taken up later when we consider volume and area.

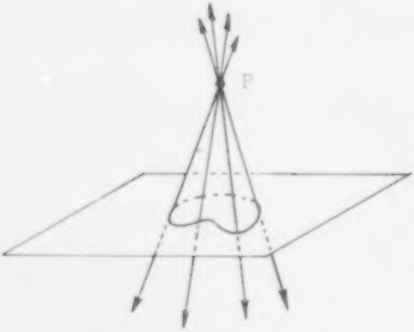
We may also treat the familiar cone and pyramid shown below as special cases of a more general classification.





Starting with a simple closed curve in a plane, a point P not in the plane, and all lines through the point and the curve, a surface is generated as shown.

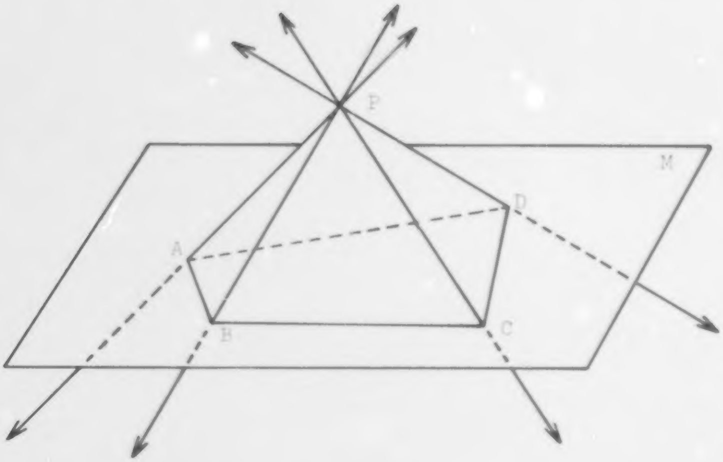




Such a surface is called a *conical surface*.

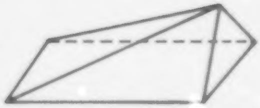
Notice that again we are using a word, in this case conical, in a more general sense than is usual in everyday language. Common usage of conical implies circular, although this is not the case with our use in mathematics. The general case above, however, certainly includes the common conception of a conical surface as a circular cone. This occurs when the simple closed curve is a circle.

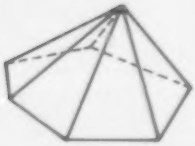
In the following sketch the simple closed curve is shown as a polygon.



Actually, the conical surface continues indefinitely in both directions. It consists of two pieces with only one point in common. These pieces are called nappes. The second nappe, not shown in the figure, occurs inverted and above the point P.

If the intersection of one of these nappes and a plane is a polygon, then the resulting closed surface is called a pyramid. In the figure above we see pyramid PABCD. The point P is called the apex if the pyramid, polygon ABCD is called the base of the pyramid. A tetrahedron is also an example of a pyramid. The familiar circular cone is formed when the intersection is not a polygon but a circle. Like prisms, pyramids are classified by their bases. The tetrahedron is a triangular pyramid. A rectangular and hexagonal pyramid are shown below.





The construction of cardboard or stiff paper models of many of the above prisms and pyramids is instructional for students and they find it very enjoyable. Some patterns are given in the SMSG Mathematics for Junior High School, Volumes I and II. These solids may be used in counting edges, vertices, and faces, in verifying Euler's formula, and are very helpful in developing space perceptions.

Class Exercises

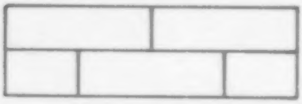
15. If a plane cuts a pyramid between the apex and the base, that portion of the pyramid which does not include the apex is called a truncated pyramid. Sketch a truncated pyramid with a hexagonal base.
16. Sketch the pattern for a triangular prism.
17. Does a cylindrical surface separate space into two subsets?
18. Does a conical surface separate space into two subsets?

11.5 Side Trips (Optional)

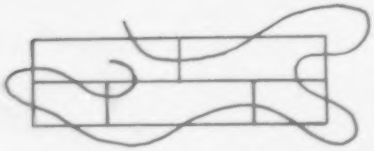
There are a variety of side trips in geometry that are non-metric in nature. Many of these are of a puzzle nature, and although they can be cast in a humorous vein, they are important on another level.

One of these problems that is related to the Koenigsberg Bridges problem is the following:

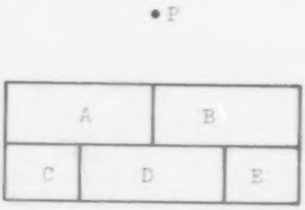
Draw a continuous line cutting each segment exactly once.

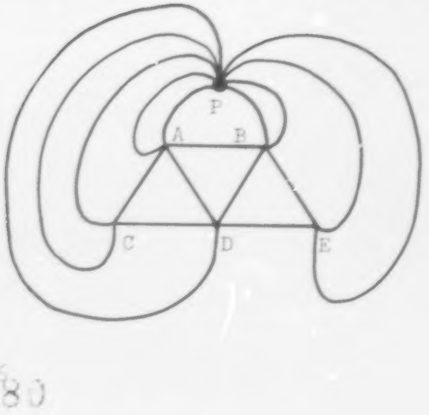


This seems to be a simple problem and indeed it is simple to state; however, its solution is elusive. A first effort such as the following,

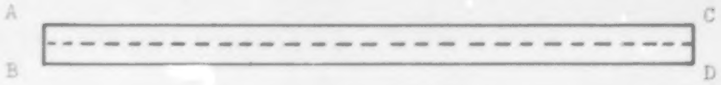


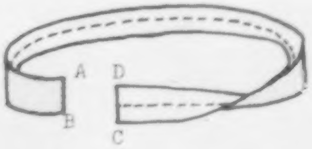
seems to need only a little change to be successful. However, such changes always seem to require other adjustments. Students find this problem very challenging and enjoy seeking a solution. Actually, a solution is not possible as may be shown by treating it as a unicursal problem. Think of each crossing of a segment as a path, and let each region shrink to a point. If we letter the enclosed regions A, B, C, D, and E, and the exterior P, then drawing the required line is equivalent to tracing the figure below without lifting your pencil and without retracing a segment. Then it becomes a network which may be examined for odd and even vertices. Points A, B, D, and P are all odd. Recalling from the introduction that no pattern with more than two odd-vertices can be traced we conclude the problem is impossible.





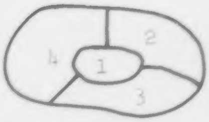
A peculiar object that contradicts many of our common notions about surfaces, and edges, is the Moebius Strip. This is made from a strip of paper, made into a loop by giving one end a twist before fastening.





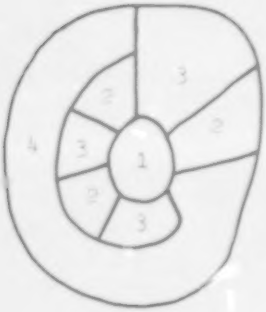
If you attempt to color "one" side of the Moebius Strip, you will discover that it has only one side! Also, following one edge will show that it has only a single edge! A still more surprising result occurs when you (or students) attempt to cut it into two pieces along the dotted line. You will also find it interesting to investigate what happens when you cut one-third of the way in from one edge. A Sunday newspaper, a roll of scotch tape, and a pair of scissors will provide many interesting questions about a Moebius Strip.

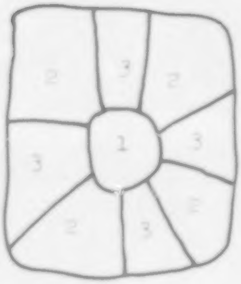
Another problem that is easy to pose and has been tackled by mathematicians without success for many years is the four-color map problem. The problem is the following: "What is the minimum number of colors necessary to color a map so that no two adjacent countries have the same color? It is easy to draw a map that will require four colors.

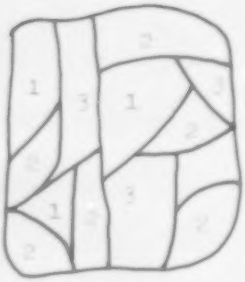


Here numerals have been used to designate colors.

It is generally believed that 4 colors are sufficient to color any map, but as yet no proof of this conjecture has been given. Neither has anyone been able to draw a map that would require more than 4 colors. Following are some maps and a way of coloring them with 4, or fewer, colors.







Junior high school students enjoy drawing such maps and attempting to color them in four colors or less. They also enjoy challenging you to color such maps. With a little practice, you can color most maps in a few minutes.

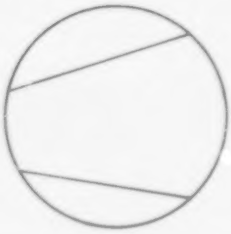
A discussion of this problem, which at present has neither proof nor disproof, provides a good opportunity to explore with students the difference between proof in general, and drawing conclusions by examining many cases. The fact that it seems possible to color all maps we may draw does not imply that we will be able to so color all maps in the future. You may also discuss the importance of a single counter example, which is sufficient to prove a statement false. Such a discussion will help to illuminate the statement attributed to Albert Einstein regarding his theory of relativity, "No number of observations will ever prove me correct: a single observation may prove me wrong."

11.6 Conclusion

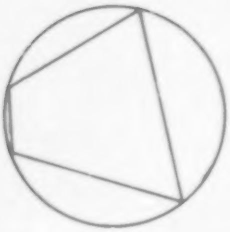
In this chapter and in the preceding one we have looked at aspects of geometry that were not dependent upon measurement or distance. Thus, many of the geometrical facts familiar to you have been omitted. We have seen no equilateral triangles, no congruent figures, no rectangles, no right angles, etc. The next two chapters, however, will consider many of these ideas.

Neither have we established a theorem-proof sequence that you undoubtedly recall from your study of geometry. Our objective in seventh grade is not to teach deductive geometry or to develop an axiomatic system, but rather to provide the students with enough background so that their formal study of the subject will proceed more easily. In grade ten or wherever formal geometry is encountered, a careful logical study of the last two chapters (and the next two) will be undertaken. At that time a careful distinction between axioms and theorems will be made. An axiom, a statement accepted as true without proof, has much the same position as an undefined word. Theorems are statements that are established as true by proof, using axioms, definitions, undefined words, and previously established theorems. They roughly correspond to our definitions made from undefined words.

It is important to consider carefully the space over which our axioms are meaningful. For example, how would our geometry differ if we limited our "space" to a circle and its interior? All our axioms could remain unchanged, but the results would be quite different. Points would still be points, but would all be located on the circle or its interior. Instead of lines extending indefinitely, they would stop at the circle. How would "rays" and "angles" differ? If "parallel" lines are still defined as non-intersecting, how do they look in our new space? What can we say about any quadrilateral with its vertices on the circle? Is it a "parallelogram?"



"parallel" lines



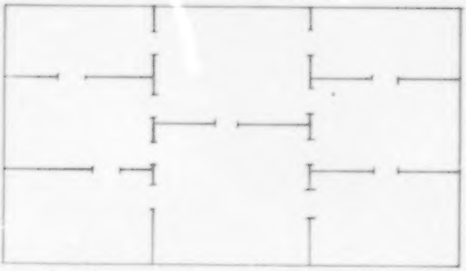
a "parallelogram"

Do every two intersecting lines form vertical angles? You might find it interesting to speculate about the differences between such a limited geometry and the one we are in the process of establishing.

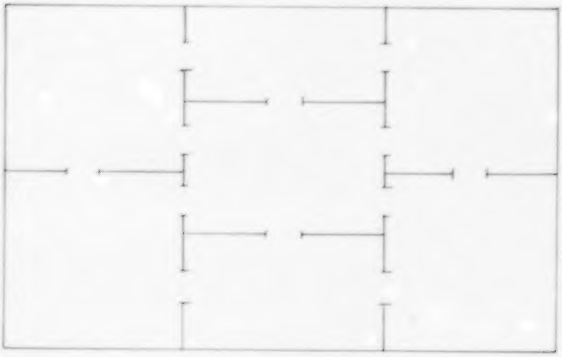
You will enjoy reading Flatland, A Romance in Two Dimensions, by E.A. Abbott. This is an interesting and amusing book describing a world of two dimensions peopled by geometrical figures. The hero is in jail for claiming to have talked to a mysterious voice from some higher dimension.

Class Exercise (Optional)

19. a. Is it possible to walk in the house with the floor plan below, passing through each door exactly once? If so, can you start in any room?

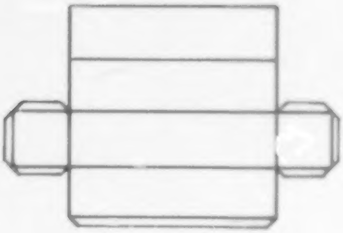


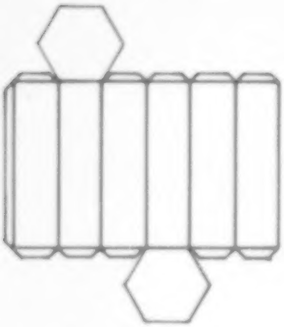
b. Answer the same questions for this plan.



Chapter Exercises

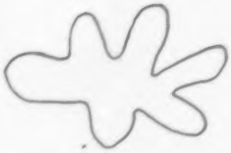
1. Redraw the patterns below on stiff paper and make models of the prisms. What are their names?



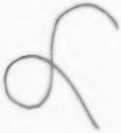


2. Which of the following are closed curves? Which are both closed and simple?

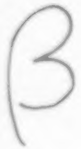
(a)



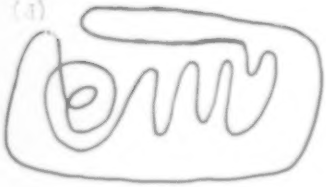
(b)



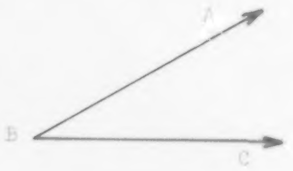
(c)

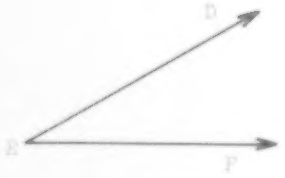


(d)



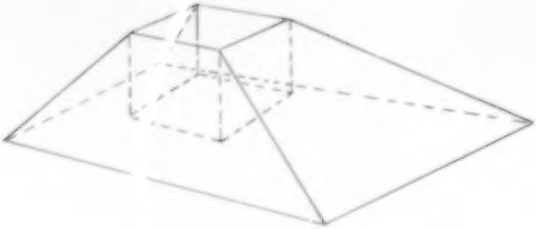
3. In the figure below is $\angle ABC \cong \angle DEF$?





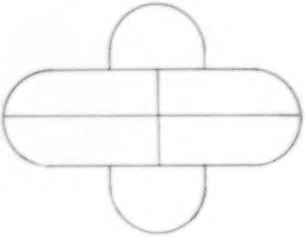
4. Draw the following:
   - a. a closed curve which is not simple.
   - b. a curve which is not closed but separates the plane into two regions.
   - c. a curve which separates the plane into three regions.

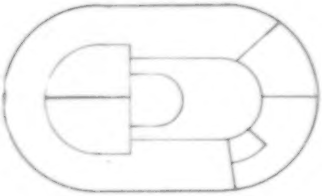
5. Does Euler's formula hold for the following solid?



6. Make a Moebius Strip with two twists instead of one and investigate its properties.

7. Color the following maps with as few colors as possible.





Answers to Class Exercises

1. The exterior of an angle is the set of all points of the plane that are neither in the interior of the angle nor on the angle.

   The exterior of a triangle is the set of all points of the plane that are neither in the interior of the triangle nor on the triangle.

   Other definitions may be given, but must be examined carefully to see whether they include or exclude any regions.

2. a. point E c. ϕ e. $\triangle$ BDE
   b. points B and C d. point B f. $\overline{AC}$ and point B

3. ($\angle$APB, $\angle$EPF): ($\angle$APD, $\angle$CPF): ($\angle$BPC, $\angle$DPE): ($\angle$APC, $\angle$DPF): (there are other pairs).

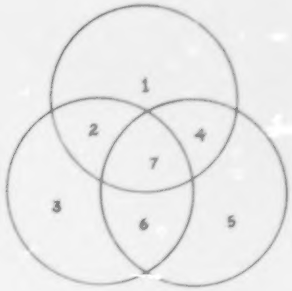
4. D - side of BE: A - side of CD: E - side of AF: (there are others).

5. $\overrightarrow{PE}$

6.

Number of Sides	Number of Diagonals
3	0
4	2
5	5
6	9
7	14
8	20
.	.
.	.
.	.
n	$\frac{1}{2}$ n (n - 3)

7. This question is intended only to provoke some thought on simple closed curves. It is interesting however to examine numbers of possible curves. From the Jordan Curve Theorem we know that any simple closed curve will bound some interior region. Thus we may examine the possible combinations of adjacent regions. Except for those that have only one point in common, every combination of adjacent regions is associated with a simple closed curve.



Thus region 1 is determined by a simple closed curve, as are the double combinations 1-2 and 1-4. The pair 1-7 is eliminated, however, since they have only one point in common and would not therefore result from a simple closed curve. The following combinations of three regions also result from such curves: 1-2-3, 1-2-7, 1-4-5, 1-4-7, 1-2-4. Analysis of this nature, taking advantage of symmetry where possible will reveal that there are 63 simple closed curves contained in the original figure.

8. The word "line" was used only with the connotation of straight, thus the term "curved line" is probably a contradiction.

9. a. the "star"
   b. the line segment, the "moon"
   c. the "dog" or the "bone"

10. P is in the exterior; Q is in the interior.

11. Before the last utility has been connected the other two have formed a simple closed curve with one house in the interior and one utility in the exterior.

12. No.

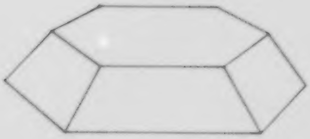
13. a. $\angle CAP$, $\angle ABE$
   $\angle EBQ$, $\angle CAB$
   $\angle PAD$, $\angle ABF$
   $\angle QBF$, $\angle BAD$

   b. $\angle CAB$, $\angle ABF$
   $\angle BAD$, $\angle ABE$

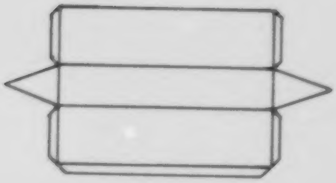
   c. $\angle CAB$, $\angle PAD$
   $\angle CAP$, $\angle BAD$
   $\angle ABE$, $\angle QBF$
   $\angle ABF$, $\angle EBQ$

14. ▱ DEMG is partially in ▱ACLG (other answers are possible).
There are nine parallelograms, three diagonals, and seven triangles.

15.

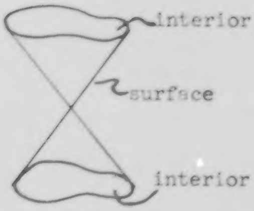


16.



17. No, three subsets: the surface, its interior, and its exterior.

18. No, four subsets: the surface itself, an "exterior", and two "interiors".



What we most commonly think of as a cone, however, does separate space into three subsets.

19. a. Yes. Any path must start in one of the rooms with five doors and end in the other.

b. Not possible, since more than two rooms have an odd number of doors.

Chapter 12

MEASUREMENT

Introduction

In the last two chapters some of the non-metric properties of certain sets of points were developed. In this and the following chapter these ideas will be related to the physical world through measurement. Historically, geometry developed through the needs of man to measure and compare certain physical things in his environment. Even the word "geometry" came from words which meant "earth measure."

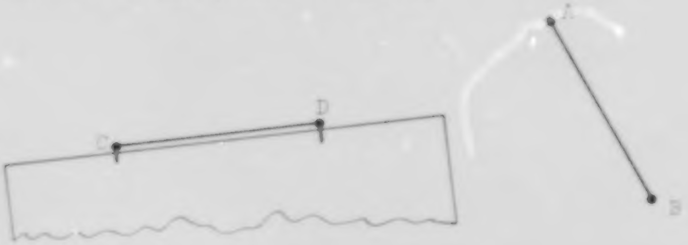
These chapters will not develop a rigorous explanation of measurement properties, but will attempt to furnish intuitive ideas of length, area, and volume concepts as presented in the SMSG Mathematics for Junior High School, Volume I. It is important that youngsters understand the approximate nature of measurement, the development of arbitrary units for measuring various physical objects, and the mathematical interpretation placed on these concepts. A major point to be made in this chapter is the fact that in measurement our units are completely arbitrary and although we are free to choose a variety of units, we ultimately settle on the most common standard units for convenience and ease of communication. Scientists are frequently confronted with measuring situations where it is more convenient to create a new unit than constantly work with very large multiples or very small parts of other units. The "light-year" and "Angstrom" are both units created to fill such special needs.

Such common questions as "How many people went to the ball game?" or "How much meat shall I buy?" or "How fast can a jet travel?" have answers which are alike in one respect: They all involve numbers. Some of these answers are found by counting, while others are found by measuring.

The question "How many?" indicates that you are thinking of a set of objects and wish to know how many there are in the set. Such a set is called a discrete set. Questions as "How much?", "How long?", "How fast?", etc., are used to describe something thought of as all in one piece, without any breaks. Such a set is called a continuous set. Sets of people, houses, or animals are discrete sets; a rope, a road, or a flagpole are all thought of as being continuous since they are like models of line segments; you can count a number of line segments but you cannot count the number of all points on a line segment. A blackboard and a pasture may be thought of as sets of points enclosed by simple closed curves and as being continuous. Such sets of points are not counted; they are measured.

12.1 Congruence

The sizes of some continuous sets may be compared in various ways. For example, to compare segments $\overline{AB}$ and $\overline{CD}$, lay the edge of a piece of paper along $\overline{CD}$ and mark points C and D.



Place the edge of the paper along $\overline{AB}$ with point C on point A. If D is between A and B, $\overline{AB}$ is longer than $\overline{CD}$. If D falls on B, the segments have the same length. If B is between C and D, $\overline{CD}$ is longer than $\overline{AB}$.

Of course, we need to recognize here that what we are really doing is idealizing this situation. It is impossible to draw representations of two line segments so that they both have exactly the same length. This again is an abstract intuitive idea that should not become entangled with the physical representations. Students should realize the differences between abstract concepts and physical interpretations of these abstractions, that the drawings they make are only to help them interpret the mathematics they study.

Let us return to the segments above and consider particularly the case where they both have the same length. When we write $4 = 2 + 2$, we mean that "4" and "2 + 2" are two names for the same number. When we write $\overline{AB} = \overline{CD}$, we mean that $\overline{AB}$ and $\overline{CD}$ are two names for the same segment; that is, the two segments are the same set of points. If $\overline{AB}$ and $\overline{CD}$ have the same length but are not the same set of points, our definition of equality does not allow us to say they are equal. They are equal only in size and shape. We use the word congruent to describe this relationship. The symbol denoting congruence is " ≅ ", and we may now write: $\overline{AB} \cong \overline{CD}$. This is read: "Segment $\overline{AB}$ is congruent to segment $\overline{CD}$." If we wish to say that the lengths of the two segments are the same, we may use the notation "AB" for the length of segment $\overline{AB}$, and write $AB = CD$. Here we mean that the length of $\overline{AB}$ is the same as the length of $\overline{CD}$. This use of the word "congruent" is an extension of the use you probably remember from high school geometry where "congruent" almost always referred to triangles. The meaning here is basically the same as with triangles and is the same meaning students will encounter when they study formal geometry. Congruent means equal in size and shape.

In working with the above segments we tacitly assumed the following properties of continuous sets of points, which, along with one more property (stated later), are the bases of measurement.

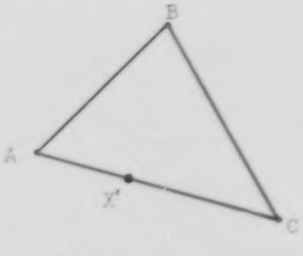
1. Motion Property. Geometric figures may be moved without changing their size or shape.

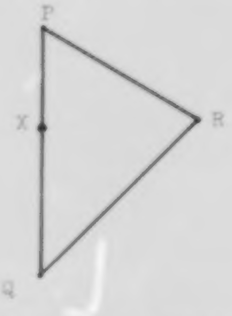
2. Comparison Property. The sizes of two geometric quantities may be compared provided these quantities have the same nature.

3. Matching Property. Two geometric quantities have the same size if every part of one can be made to coincide to a part of the second so that no part of either figure is omitted.

These are some of the properties that enable the students to relate the abstractions of geometry to the physical world, and we should be aware that these need to be pointed out to them as the measuring process is utilized.

In elementary school mathematics, students make models and tracings of geometric figures and test for congruence by determining whether two figures have the same "size and shape" by superimposition. In junior high school the ground work is being laid for a more formal definition of congruence that will occur in a deductive geometry course in high school. For example, two spheres may be "congruent", but imposing one sphere on another doesn't make much sense. From the idea of superimposition let us try to move to a more formal definition of congruence.





Suppose △ PRQ can be superimposed on △ ABC with R falling on B, P on A, and Q on C. Then there exists a one-to-one correspondence between △ PRQ and △ ABC, each point of △ PRQ corresponding to that point of △ ABC which it "covers" when △ PRQ is superimposed on △ ABC. For example, the point X would correspond to the point X' under this correspondence. But it is not enough simply to say that there exists a one-to-one correspondence between △ PRQ and △ ABC. Something else is also involved in the notion of congruence. *Distances must be preserved*. Suppose $\triangle PRQ$ is superimposed on $\triangle ABC$ as indicated by the following diagram.

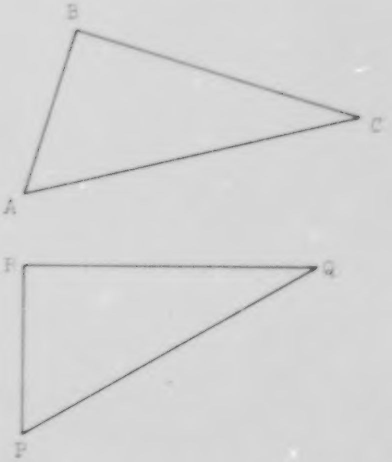
$$\begin{array}{c} P \longleftrightarrow A \\ R \longleftrightarrow B \\ Q \longleftrightarrow C \end{array}$$

(Note: The double-headed arrow shows that vertex P of $\triangle PRQ$ corresponds to the vertex A of $\triangle ABC$, and that A corresponds to P, etc.)

Then for *any* two points of $\triangle PRQ$, the distance between them (i.e., the length of the segment joining them) must be the same as the distance between the two points of $\triangle ABC$ to which they correspond. As examples, the distance between R and X must be the same as the distance between B and X' ($RX = BX'$), the distance between Q and P must be the same as that between C and A ($QP = CA$). These considerations lead us now to our definition:

> Two sets of points are said to be *congruent* provided that there is a one-to-one correspondence between them that preserves distance.

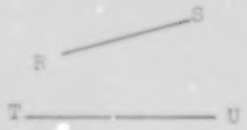
By naming our triangles carefully, we can see immediately the corresponding parts. Again considering the two triangles in the figure, we may show the correspondence as follows:



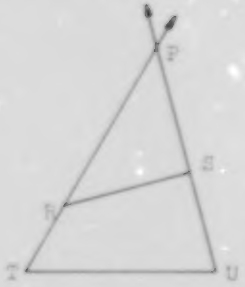
Given: $\triangle ABC \cong \triangle PRQ$

$$\begin{array}{ccc} A & \longleftrightarrow & P \\ B & \longleftrightarrow & R \\ C & \longleftrightarrow & Q \\ \overline{AB} & \cong & \overline{PR} \\ \overline{AC} & \cong & \overline{PQ} \\ \overline{BC} & \cong & \overline{RQ} \\ \angle A & \cong & \angle P \\ \angle B & \cong & \angle R \\ \angle C & \cong & \angle Q \end{array}$$

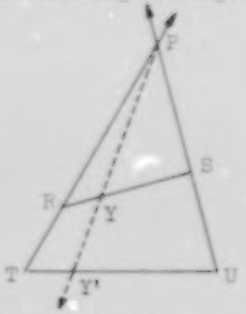
The importance of the preservation of distance for congruence needs to be stressed because it is possible to establish a one-to-one correspondence between two sets of points that does not preserve distance. For example, $\overline{RS}$ and $\overline{TU}$ below may be put into a one-to-one correspondence in the following manner:



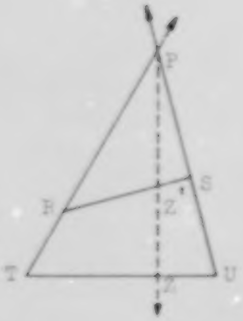
Draw $\overrightarrow{TR}$ and $\overrightarrow{US}$ and call the intersection of these rays P, as in the figure below.



Now for any point Y on $\overline{RS}$ a corresponding point Y' may be found by drawing $\overrightarrow{PY}$, as in the following drawing.

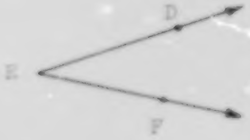


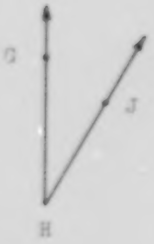
Likewise, for any point Z on $\overline{TU}$, a corresponding point Z' on $\overline{RS}$ may be located as the intersection of $\overline{ZP}$ and $\overline{RS}$. This is shown in the following drawing.



This shows that for each point on one line there is a unique point on the other line, and vice versa. Therefore, a one-to-one correspondence between all the points on $\overline{RS}$ and all the points on $\overline{TU}$ has been established, even though distance has not been preserved.

With two congruent angles it is possible to set up more than one correspondence. Given, $\angle DEF \cong \angle GHJ$,



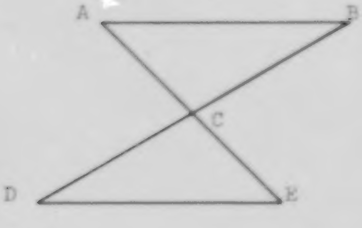


we see that $\angle DEF \cong \angle GHJ$ or $\angle DEF \cong \angle JHG$. Remember, as long as the middle letter names the vertex, the order of the letters for naming an angle is immaterial. Also we have not said that $\angle DEF$ equals $\angle GHJ$. If we do this, then we are probably talking about the measures of these angles as being the same number and will show this as $m(\angle DEF) = m(\angle GHJ)$, where "$m(\angle DEF)$" is a number indicating the measure of the angle. Here, as in segments, we are making a distinction between the angle and its measure. Even though we have not discussed "measuring" angles yet, we probably have assumed the following statement and its converse: "If two angles are congruent, then their measures are equal."

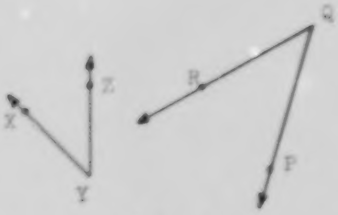
Our definition of congruence is a more sophisticated idea than we would expect seventh graders to accept, but it is the idea for which teachers of these students are laying the groundwork. By cutting, superimposing, measuring, and comparing various models of sets of points, students discover certain characteristics of segments, lines, angles, polygons, and solids. We will consider some of these in this chapter and the next.

Class Exercises

1. Given the figure with the two triangles congruent, list the corresponding parts.



2. How would you test whether $\angle XYZ$ is congruent to $\angle PQR$? Does congruence of angles depend on the length of the sides of the angles? Explain.



3. If the three sides of one triangle are congruent respectively to the three sides of another triangle, do you think the two triangles are congruent? Explain your reasoning.

4. If the three angles of one triangle are congruent respectively to the three angles of another triangle, do you think the two triangles are congruent? Explain your reasoning.

12.2 The Nature of Measurement

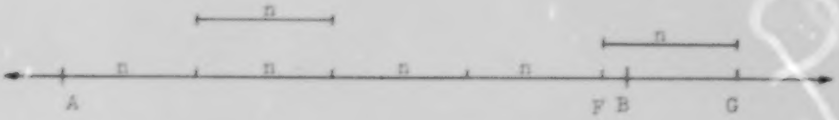
We have said that there are some sets of points, called continuous sets, which require measuring and for which counting as such is inappropriate. Answers may be given in terms of whole numbers or they may involve rational numbers, but these answers are not absolutely precise. The accuracy of the number used in the physical measurement process is restricted by unevenness in the object measured, by the measuring instrument we use, and by our own approximation to an answer. Therefore, we say that all measurement of physical objects is approximate.

We used the motion, comparison, and matching properties from Section 1 to develop an intuitive idea of congruence. These three properties, together with a fourth, the Subdivision Property, are the basis for measurement.

> 4. Subdivision Property. A geometric continuous figure or set may be subdivided.

If a segment $\overline{AB}$ is subdivided by a point C so that $\overline{AC} \cong \overline{CB}$, then the length of $\overline{AC}$ is one half the length of $\overline{AB}$.

$\overline{AB}$ may be subdivided in other ways so as to compare the length of one segment with the length of another segment. Suppose a segment is chosen of any length less than the length of $\overline{AB}$; call the length of the segment: "n".



Beginning at a point A in the figure above, a segment of length n is marked off 4 times so that $\overline{AF}$ is of length $4n$. The symbol "$4n$" means "four times as long as the segment of length n." It is said that the length of $\overline{AB}$ is approximately equal to $4n$, rather than to $5n$, because B falls closer to F than to G. A symbol for the words "is approximately equal to" is a wavy equal sign like this: " $\approx$ ". We may write in symbols $AB \approx 4n$ and read it as: "The length of segment AB is approximately equal to $4n$."

Notice how these symbols are used:

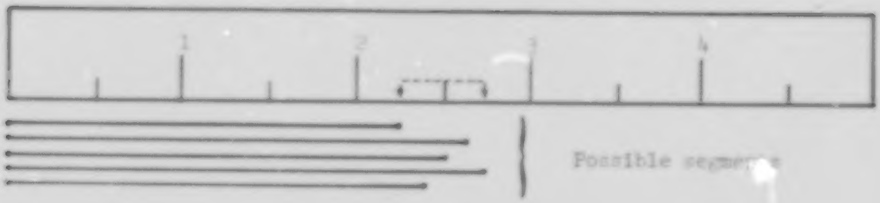
4 is the measure,
n is the unit of measurement,
$4n$ is the length.

In the example above, we picked an arbitrary unit "n" which we used to measure $\overline{AB}$, but we could have used any unit of length. Man first began comparing and measuring physical objects by choosing some convenient unit. Often this was some part of his body and was quite satisfactory for his primitive culture. But as tribes and countries began to trade with each other, a need for more standardized units became necessary, and the head of a country might decree that the "standard unit" would be "the distance from the tip of his nose to the tip of his middle finger," or some such unit as this. Even today the system commonly used in English-speaking countries is based on these primitive body measures.

In the past, disagreements about linear units became so common that a group of French scientists with representatives from many countries established an international set of measures. This group developed the metric system which discarded the old units and based all units on the distance from the North Pole to the equator. The meter is the basic unit of length in the metric system. (The meter was planned to be one ten-millionth of the distance along a meridian from the North Pole to the equator, but recently an international congress of scientists defined the meter in relation to the wave length of a certain color of light.) The metric system is used by most scientists in the world and is in common use in all countries except those in which English is the main language spoken. We will consider the metric system in more detail in the next chapter. However, the history of measurement is interesting to junior high students and can be correlated with social study units quite effectively.

Another aspect to be considered is what is meant when we say an object is 6 feet long. We will adopt the convention that we mean the length is closer to this number than to any other comparable one. In other words, we say that the object is closer to 6' than to 5' or to 7'; that the "true" length is between 5.5' and 6.5'. The greatest possible difference between the asserted length and the "true" length is not more than one-half the unit used for measuring (in this case, $\frac{1}{2}$ foot). This one-half unit is called the *greatest possible error*.

As another example, assume a measurement is given as $2\frac{1}{2}$ ", measured to the nearest half inch. The real length then lies between $2\frac{1}{4}$ " and $2\frac{3}{4}$ ", and the greatest possible error is $\frac{1}{2}$ of $\frac{1}{2}$ ", or $\frac{1}{4}$ ". A diagram may be helpful here. Note that we say the length of each of the segments below is $2\frac{1}{2}$ inches, *when measured to the nearest half inch*.



Sometimes the form, $2\frac{1}{2} \pm \frac{1}{4}$ in., is used where the "$\pm \frac{1}{4}$" indicates the greatest possible error. This shows that the object was measured to the nearest $\frac{1}{2}$ inch. Another way to write this would be $2\frac{2}{4}$ ", not changing the fraction to lowest terms, although the first method is usually preferred.

The *precision* in any measurement is shown by naming the smallest unit used. Thus in the example above the measurement is made with a precision of one-half inch, or is precise to the nearest one-half inch. Greatest possible error, however, is the greatest possible difference between the real length of a segment and the measurement stated. *Greater precision* is obtained by using an instrument whose units are subdivided by fractions with greater *denominators*. Measurements made with a ruler marked in eighths are more precise than those made with a ruler marked in fourths. A micrometer is an example of a precision instrument whose subdivisions are named by fractions with denominators of 100, 1000, and 10,000. Constant efforts to develop better precision instruments are being made by industry because of the increasing need for very close tolerances.

We can see some of the ramifications of precision and greatest possible error when we use measurements in various computations. Let us say that we have two line segments, both measured to the nearest $\frac{1}{4}$ inch: $2\frac{3}{4} \pm \frac{1}{8}$ inches and $4\frac{1}{4} \pm \frac{1}{8}$ inches, respectively. We would like to find the sum of the measurements or the length of the two segments when placed end to end. We could lay these segments end to end and measure them, but suppose we decide to add the numbers $2\frac{3}{4}$ and $4\frac{1}{4}$. We have made some computations revealing the greatest possible error of the sum:

Least Value	Reported Measure	Greatest Value
$2\frac{5}{8}$	$2\frac{3}{4}$	$2\frac{7}{8}$
$4\frac{1}{8}$	$4\frac{1}{4}$	$4\frac{3}{8}$
$6\frac{3}{4}$	7	$7\frac{1}{4}$

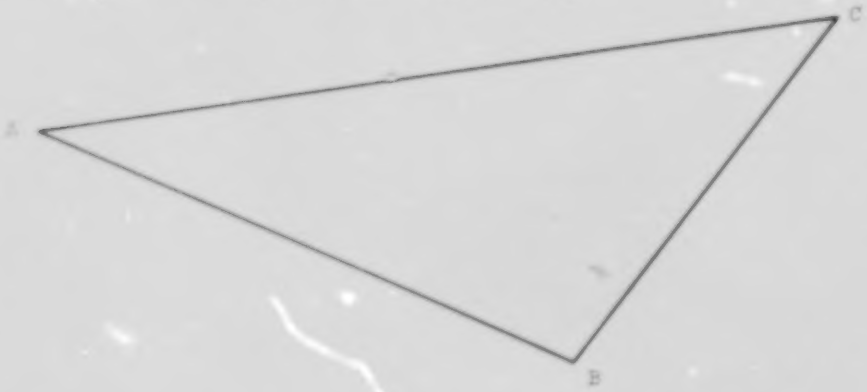
Thus the sum 7 really has the greatest possible error of $\frac{1}{4}$ and is not as precise as our original measurements. A further discussion of computation with approximate data will be found in the next chapter.

Finally we should note that some problems of measurement are psychological in nature. For example, what does a youngster mean when he says that his age is 12? What does a woman mean when she says that she is 37 years old?

Class Exercises

5. If a length is reported as $5\frac{2}{4}$ inches, the true length must be between _____ and _____. The greatest possible error is _____.

6. a. Measure the lengths of each side of the triangle to the nearest 16th inch and express your answer in two ways.



b. Add the numbers representing the measures and indicate the greatest possible error of this sum.

7. Indicate the measure and the unit for each of the following measurements.

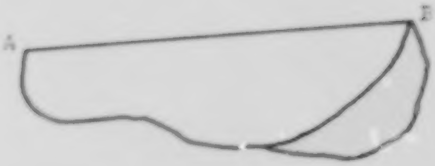
a. 3 feet
b. 17 pounds
c. 24 hours
d. 16 ounces

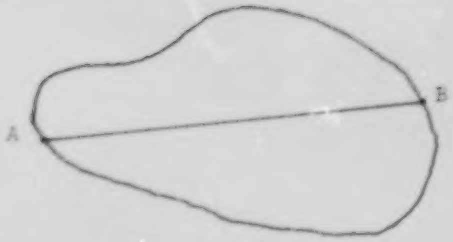
12.3 Angular Measure

Let us recall the definition of an angle: given two different rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ not on the same line, with common endpoint A, $\overrightarrow{AB} \cup \overrightarrow{AC} = \angle BAC$. We need to devise a method for measuring an angle, and we will attack this essentially as we did measurement of segments. That is, (1) the unit for measuring a segment had to be a segment; (2) the segment to be measured was compared with unit segments; and (3) the measure of the segment was the number of unit segments into which it was subdivided. Similarly, we need a unit angle with which to compare the angle to be measured. The measure of an angle is associated with its interior. To measure an angle, its interior is subdivided by the unit angle.

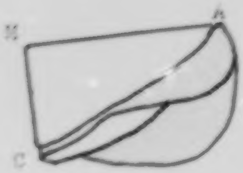
Students can select some arbitrary unit angle and in measuring various angles can review again many of the ideas of approximation in measurement. An easily obtained and simple unit angle to use is formed by folding a piece of paper as follows:

Fold it once to make a model of a line separating two half-planes. Call it $\overleftrightarrow{AB}$.



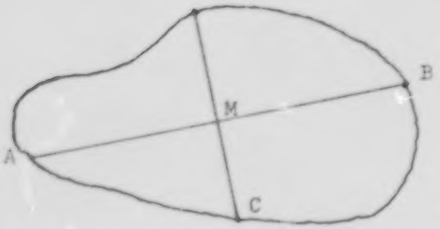


Choose a point M on $\overleftrightarrow{AB}$ and fold through M so that $\overleftrightarrow{MA}$ falls on $\overleftrightarrow{MB}$.



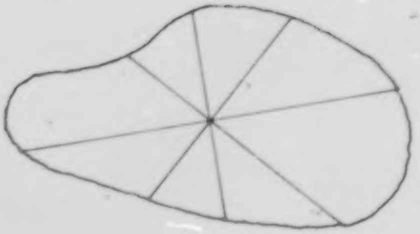
Then $\angle CMA$ is a model of a right angle.

If you unfold the paper, it will appear like this.



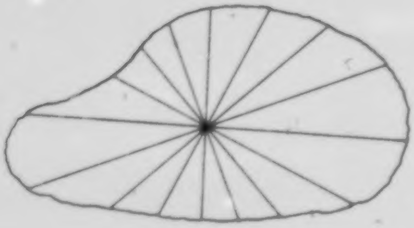
This shows four models of angles, all congruent, that together with their interiors fill the plane.

Refold the paper so that you again have a model of a single right angle. Now fold so that the rays represented by $\overline{AM}$ and $\overline{CM}$ coincide.



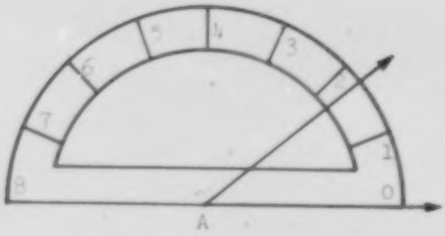
This provides us with a model of an angle such that any four successive angles with a common vertex will exactly fit in the half-plane.

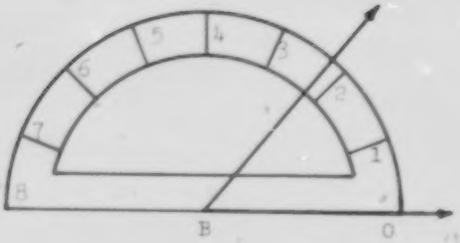
Refold your paper. Proceed to make one more fold as before. You now have a model of an angle, eight of which, successively placed with a common vertex, will exactly fit on the half-plane and its edge. The picture below shows a model of sixteen such angles. Since this is not a common unit, we might call it an "octon," since eight fill a half-plane.



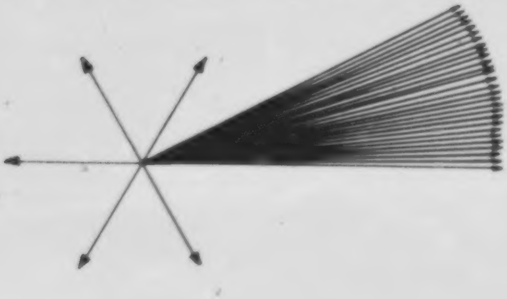
We may use eight of these octons as a simple protractor. Each ray of the successively marked-off octons may be associated with a whole number, taken in order from 0 to 8, to give a protractor with a scale on it suitable for use in measuring angles. It should be emphasized that the measure of an angle is a number. We read "m(∠ABC) = 7" as "The measure of an angle ABC is seven." This statement of equality is permissible since the measure of angle ABC is a number.

Eventually the pupil recognizes that approximate readings of angle measures "to the nearest octon" lead him into a situation such as shown below in which both ∠A and ∠B (clearly not the same size) have a measure of 2, to the nearest octon.





The need for a smaller unit soon becomes apparent. The standard unit of angle measure most commonly used is the degree. Other units are used in more advanced or specialized work but will not be discussed here. The degree may be determined by a set of rays drawn from the same point on a line such that they determine 180 congruent angles. These 180 angles with their interior form a half-plane and its boundary, the line. Each of these angles is a standard unit angle. Its measurement is called one degree, and we write it 1°. When we speak of the size of an angle, we may say its size is 45°. However, if we wish to indicate the measure of the angle, we must realize that a measure is a number and say that its measure, in degrees, is 45. If we lay off 360 of these unit angles, using a single point as a common vertex, then these angles together with their interiors cover the entire plane.



Even in ancient Mesopotamian civilization the angle of 1° as the angle of unit measure was used. The selection of a unit angle which could be fitted into the plane just 360 times, (as above), was probably influenced by their calculation of the number of days in a year as 360. In this book we concern ourselves only with angles whose measures are between 0 and 180. Because of our definition of an angle, it is not possible to have an angle whose rays coincide or extend in a straight line.

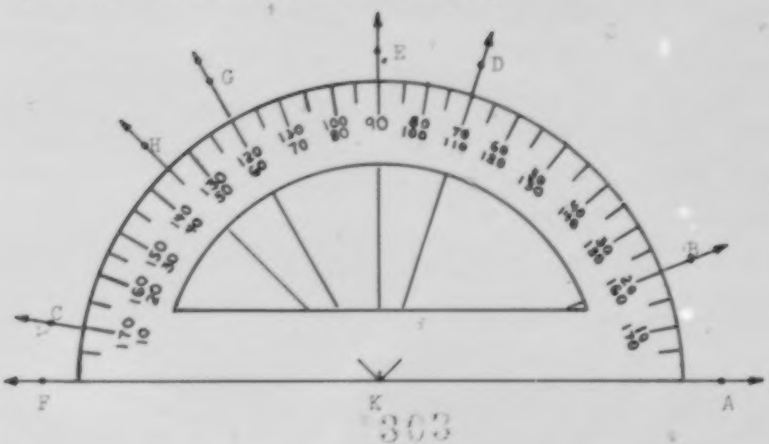
Thus the measurement of angles essentially becomes a process of determining how many times the given unit angle is contained in the given angle. What we are assuming, of course, is the existence of a one-to-one correspondence between all angles and all the numbers between 0 and 180. In fact, this very one-to-one correspondence is postulated in many new geometry books. The correspondence is similar to the one-to-one correspondence between all points on a line and all real numbers.

Remember that measurement is only approximate, and often it is difficult for youngsters to draw and measure angles precise to 1°. The markings on a standard protractor are closely spaced, and the width of the side of a model of an angle may fill the space between two of these markings. Therefore, when a measurement of an $\angle ABC$ is given as 65 degrees, it should be indicated as: $m(\angle ABC) \approx 65$. Protractors of clear plastic are available and are quite effective for demonstrations on the overhead projector.

An exercise that students can do is to draw several angles, then find the measures, in "octons," of these angles. Using a protractor, the measures, in degrees, may also be found. Students also like to exchange papers and measure the angles their classmates have drawn.

Class Exercises

8. The sketch shows a protractor placed on a set of rays from point K. Find the measure, in degrees, of each angle named.



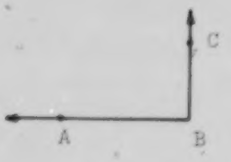
a. $\angle AKB$
b. $\angle FKE$
c. $\angle AKC$
d. $\angle FKG$
e. $\angle AKD$
f. $\angle BKE$
g. $\angle CKD$
h. $\angle HKD$
i. $\angle DKB$
j. $\angle HKC$

12.4 Classification of Angles and Triangles

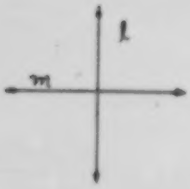
Now that we are more familiar with congruence and linear and angular measure, let us explore some geometrical facts related to ideas of distance and measurement. This section states many definitions already familiar to you but are given here for your reference. Seventh grade students sometimes encounter difficulties in visualizing all of the cases of a particular definition. We will attempt to point out some of these trouble spots in this section. Again, however, students need to have an intuitive feeling for the ideas presented here before they can verbalize them meaningfully.

We may now define a right angle as an angle whose measurement is 90 degrees, one whose size is less than 90 degrees as an acute angle, and one whose measure is more than 90 degrees as an obtuse angle. Notice that because the measure of an angle is associated with its interior, we do not need to say that an obtuse angle has a degree measure of less than 180.

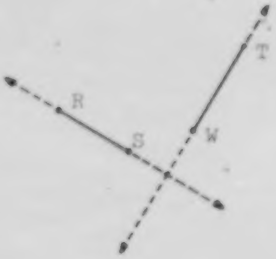
When two lines intersect, they are perpendicular (symbol: $\perp$) if one of the angles determined by the lines is a right angle. Line segments and rays are said to be perpendicular if the lines containing them are perpendicular. Observe several of the possibilities below. Students sometimes do not want to accept the conditions as displayed in (c) and (e). Two pieces of wire, or even pencils, representing segments, placed on the stage of an overhead projector, will often help to make this clear.



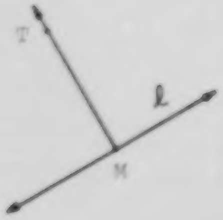
(a) $\overrightarrow{BA} \perp \overrightarrow{BC}$



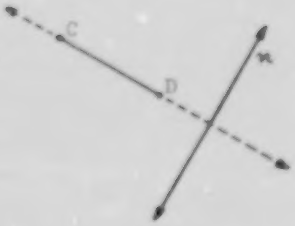
(b) Line ℓ $\perp$ line m



(c) $\overline{RS} \perp \overline{TW}$



(d) Line ℓ $\perp$ $\overrightarrow{MT}$



(e) $\overline{CD}$ $\perp$ line n

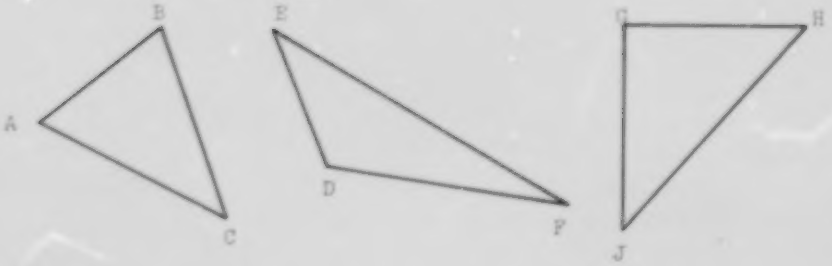
If two angles have the same vertex and have a common ray but have no interior points in common, then they are called adjacent angles. If the sum of the measures, in degrees, of two angles is 180, then the angles are called supplementary angles. If the sum of the measures, in degrees, of two angles is 90, then they are complementary angles. Supplementary and complementary angles may be adjacent but this is not necessary. Again, these two terms are often confused, and students need to see many instances of both before the definitions are well established in their minds. The English usage of the two words (as well as the word "complement") is also a little different than the mathematical usage, and this may need to be pointed out.

Class Exercises

9. If two adjacent angles are supplementary, what can you say about the line formed by the "outside" rays?

10. If two adjacent angles are complementary, what can you say about the "outside" rays?

Again for your reference, we may classify triangles according to either their sides or their angles.



In triangle ABC all the angles are acute angles, and $\triangle ABC$ is called an acute triangle; also $\triangle EDF$ with the obtuse angle EDF is called an obtuse triangle. One of the angles in $\triangle GHJ$ is a right angle and the triangle is called a right triangle.

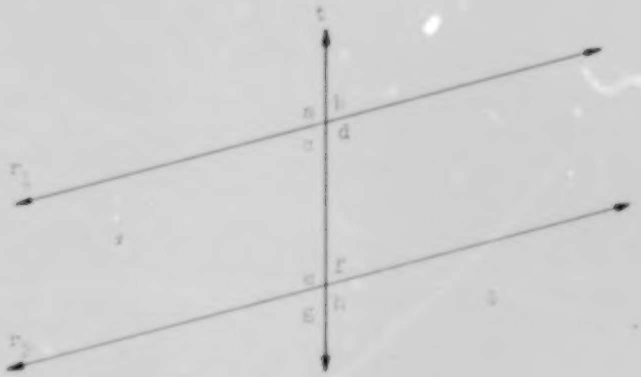
Using sides of a triangle for classification, we say that if none of the sides of a triangle are congruent, then the triangle is scalene. If two sides are congruent, then the triangle is isosceles. If all three sides are congruent, then it is equilateral.

Some of the following exercises are examples of trouble spots for students, but they often enjoy trying to find a counter-example. The converse of a conditional statement may cause difficulties (see Exercise 16), but here is a place where logical reasoning may be stressed to good advantage.

Class Exercises

11. Is it possible to have
    a. a scalene right triangle?
    b. an isosceles right triangle?
    c. an equilateral right triangle?
    d. an isosceles obtuse triangle?
    e. an equilateral obtuse triangle?
12. What seems to be true of the angles of an equilateral triangle?
13. What seems to be true of two of the angles of an isosceles triangle?
14. If a triangle is equilateral, is it also isosceles?
15. Is the converse of the statement in Exercise 14 true?

In Chapter 11, names were given to certain pairs of angles formed when two lines are cut by a transversal, namely, corresponding angles and alternate interior angles. The SMSG text, Mathematics for Junior High School, Volume I, very effectively leads students through a discovery of the relationship between corresponding angles and shows that when parallel lines are cut by a transversal, the corresponding angles are congruent.



In the figure above, r_1 and r_2 are parallel (i.e., $r_1 \cap r_2 = \emptyset$), and t is a transversal. .

The two angles in each pair of corresponding angles are congruent and hence equal in measure. Thus, we may write:

$\angle a \cong \angle e$ $\qquad$ $m(\angle a) = m(\angle e)$

$\angle b \cong \angle f$ $\qquad$ $m(\angle b) = m(\angle f)$

$\angle c \cong \angle g$ $\qquad$ $m(\angle c) = m(\angle g)$

$\angle d \cong \angle h$ $\qquad$ $m(\angle d) = m(\angle h)$

We will not go through this development but will list this property and two others which will be used in a subsequent geometric proof.

I. Vertical angles formed by two intersecting lines are congruent.

II. Two lines in the same plane and intersected by a transversal are parallel if and only if a pair of corresponding angles are congruent.

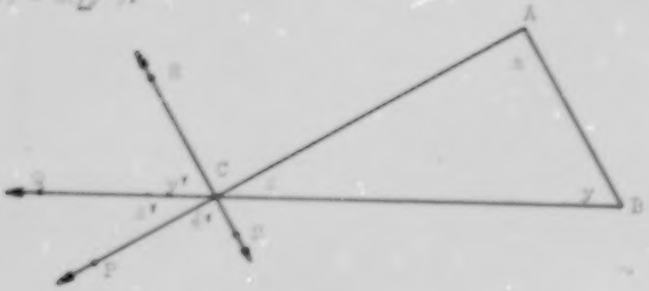
Let us now prove, through a class exercise, the following statement about triangles:

> The sum of the measures, in degrees,
> of the angles of any triangle is 180.

The proof is based on the property that if a set of angles and their interiors form a half-plane and its boundary, then the sum of the measures of the angles is 180.

Class Exercise

16. Consider the $\triangle ABC$ and $\overrightarrow{AP}$ and $\overrightarrow{BQ}$. $\overleftrightarrow{RS}$ is drawn through point C so that $m(\angle y) = m(\angle y')$.



Answer the questions and use a property to explain "why" for each of the following:

a. Is $\overleftrightarrow{RS}$ parallel to $\overline{AB}$? Why?

b. What name is given to the pair of angles marked x and x'? Is $m(\angle x) = m(\angle x')$? Why?

c. What name is given to the pair of angles marked z and z'? Is $m(\angle z) = m(\angle z')$? Why?

d. $m(\angle y) = m(\angle y')$ Why?

e. $m(\angle x) + m(\angle y) + m(\angle z) = m(\angle x') + m(\angle y') + m(\angle z')$ Why?

f. $m(\angle x) + m(\angle y) + m(\angle z)$ is the sum of the measures of the angles of the triangle. Why?

g. $m(\angle x') + m(\angle y') + m(\angle z') = 180$ Why?

h. $m(\angle x) + m(\angle y) + m(\angle z) = 180$ Why?

i. We conclude therefore that the sum of the measures, in degrees, of the angles of the triangle is 180.

A formal proof of a geometric theorem, usually appearing in tenth year geometry texts, has just been developed. However, it is important to note that this should not be done with seventh graders unless a great deal of groundwork is laid and an intuitive development of these properties has occurred. Students need to measure and find the sums of the measures of the angles of many triangles. They should cut off two angles of a paper model of a triangular region and place them beside the third angle and see that the three angles and their interiors seem to fill the half-plane. Also, before these properties can be used as reasons in a proof, the pupils have to state them in precise mathematical language and understand fully what they mean.

In this section we have not stated many of the properties of geometric figures, and we have not given a definition of many of the common polygons. Some of these are left for you as class exercises and chapter problems. As with much of the mathematics presented at the junior high level, geometric concepts can best be developed by having students use paper and pencil as they read and listen, by letting them construct models, and by the teacher asking leading questions. On the other hand, much of mathematics is quite abstract, and the students need to be led toward these abstractions as they progress through the junior high school years.

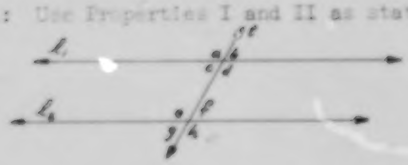
Class Exercises

17. Given a line r and a point P not on the line, define the shortest segment from P to r.

18. Define what you think is meant by the distance between two parallel lines.

19. In Chapter 11 a parallelogram was defined, but a rectangle could not be defined. Why not?

20. Prove: If two parallel lines, ℓ_1 and ℓ_2, are intersected by a transversal, t, then a pair of alternate interior angles, $\angle c$ and $\angle f$, are congruent. (Hint: Use Properties I and II as stated in this section.)



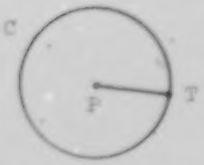
12.5 Circles

One of the most common simple closed curves is the circle, yet in the chapters on nonmetric geometry we were not able to give a definition of a circle. Why not? The reason is that we need the concept of distance and measurement to define a circle. From the primitive idea that a circle is "round," through the idea that it is the set of points at a fixed distance from a given point, students may develop the following definition.

> A circle is a simple closed curve in a plane, each of whose points is the same distance from a fixed point in the same plane called the center.

May we repeat again that the definitions stated in this section are included only for completeness and handy reference. However, some of these might refresh your memory, as they certainly are new to many of the more recent junior high school programs.

In the figure below, point P is called the center; but, by definition, the center is not part of the circle. The segment $\overline{PT}$ is called a radius of the circle and is defined as any segment which joins the center P to a point on the circle. The word "radius" is sometimes used to mean the distance from the center to any point on the circle. Usage will generally indicate the correct interpretation for the word.

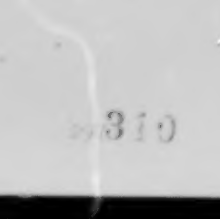


A diameter of a circle is a segment that contains the center of the circle and whose endpoints lie on the circle. The relationship between the radius and the diameter of a circle can be expressed as:

$$d = 2r, \quad \text{or} \quad r = \frac{1}{2}d.$$

This seems like a trivial relationship, but it is important a little later in our development of areas of circular closed regions.

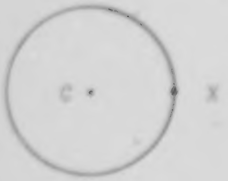
Certain other sets of points often associated with a circle may be mentioned.



In the figure above, line ℓ contains exactly one point of the circle and is called a tangent to circle P. The intersection of the circle and the tangent is point T, called the point of tangency. The endpoints of segment $\overline{AB}$ are on the circle, and $\overline{AB}$ is said to be a chord of the circle. By this definition is a diameter also a chord?

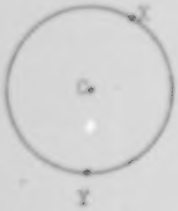
In Chapter 10, separations of lines, planes, and space were discussed. A point separates a line into three subsets: the two half-lines and the point itself. A line separates a plane into three subsets: the two half-planes and the set of points on the line. Describe how a plane separates space into three subsets.

Does a circle separate a plane into three subsets? Yes, the three sets are the interior region, the set of points on the circle itself, and the exterior region. Does a single point on a circle separate the circle into three subsets? Does point X, for example, separate the circle below into three subsets?



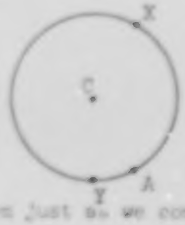
We see that whether we move in a clockwise or a counterclockwise direction, we will eventually return to X. Therefore, a single point separates a circle into only two subsets, unlike the situation with the line.

Just as we considered parts of lines called line segments, we will consider parts of circles called arcs.

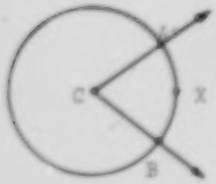


In the drawing above, the circle is separated into four parts, or subsets: the two points X and Y and the two arcs determined by them. If no ambiguity results, we usually consider the "shorter" of the two arcs and name it $\widehat{XY}$.

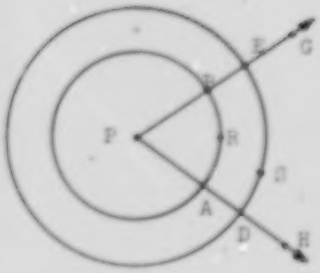
The symbol "$\frown$" represents the word "arc." When the possibility of confusion exists, we label a point on the arc as in the figure to the right. We may now speak of $\widehat{XAY}$ without ambiguity.



In working with arcs we often wish to compare them just as we compare lengths of line segments or measures of angles. Think of a circle divided into 360 congruent arcs. Each such arc determines a unit of arc measure called one degree of arc. Rays from the center of the circle, passing through the endpoints of an arc, determine a central angle. We may think of a degree of arc as being determined by a central angle which is a unit angle of one degree.



In the figure above, if the measure of central angle $\angle ACB$, in degrees, is 70, then the measure of $\widehat{AXB}$ in degrees is also 70, written: $m(\widehat{AXB}) = 70$. Remember that arc measure is not a measure of length. For example, consider the two concentric circles below:



The two arcs $\widehat{ARB}$ and $\widehat{DSE}$ have the same central angle, $\angle GPH$. Therefore, $\widehat{ARB}$ and $\widehat{DSE}$ must have the same arc measure, even though the "length" of $\widehat{ARB}$ is shorter than the "length" of $\widehat{DSE}$. The length of a circle is called the circumference and this will be discussed in the next chapter.

One note of caution: Some students have difficulty using rulers, protractors, and compasses for drawing figures and measuring. Although mathematics is not a course in which drafting should be taught, it is essential that students receive some instruction and practice in the use of these devices.

Class Exercises

21. Two meanings were given to the word "radius." What are the two meanings of the word "diameter?"

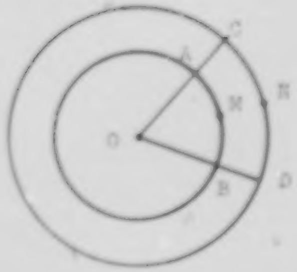
22. Define diameter in terms of a chord.

23. Draw a circle and an angle in the same plane so that their intersection consists of: a. 1 point, b. 2 points, c. 3 points, d. 4 points, e. no points.

24. Describe the interior of a circle using the concept of distance.

25. How many degrees in a quarter of a circle? In one-eighth of a circle? In five-sixths of a circle?

26. Given two concentric circles, demonstrate a one-to-one correspondence between the points in $\widehat{AMB}$ and the points in $\widehat{CND}$.



12.6 Conclusion

This chapter has attempted to extend nonmetric geometry by developing the concepts of congruence, the nature of measurement, and a brief discussion of circles. In the next chapter we will continue this discussion on the metric properties of sets of points by examining perimeters, areas, volumes, and systems of measures.

Sometimes the intuitive and measurement aspects of geometry become bogged down in a dictionary approach. It is important that this be avoided. Students develop nonverbal awareness of many of these ideas before they can state them formally. Through discovery they see relationships in sets of points, and their interest and enjoyment in understanding this kind of material is aroused.

Chapter Exercises

1. Draw a segment 2 inches long and divide it so that it can be used as a ruler to show a precision of one-eighth inch.

2. Draw a segment 2 inches long and divide it so it can be used as a ruler to show a greatest possible error of one-eighth inch.

3. A rectangle has a length of 5 inches and a width of $3\frac{1}{2}$ inches. Each measurement is given with a precision of $\frac{1}{2}$ inch.

    a. Draw a rectangle using the longest possible segments that have these measurements.

    b. In the interior of the rectangle in (a) draw another rectangle that has the shortest possible segments with these measurements.

4. Name as many special kinds of quadrilaterals as you can.

5. What do you think is meant by a regular polygon?

6. What condition(s) are necessary and sufficient for two circles to be congruent?

7. Given a circle and a tangent to the circle. What do you think the relationship is between the tangent and the line which joins the center of the circle to the point of tangency?

8. Draw two arcs whose degree measures are each 60 but such that one seems to be twice the length of the other. What seems to be true about the radii of the circles that contain these arcs?

9. Define a sphere.

10. We proved that the sum of the degree measures of the angles of a triangle was 180. If a "triangle" is drawn on the surface of a sphere, is this still true? Give a definition of a "triangle on a sphere." What is a "right triangle" on a sphere?

Answers for Class Exercises

1. $\triangle ABC \cong \triangle EDC$

$\overline{AB} \cong \overline{ED}$ $\quad\quad$ $\angle A \cong \angle E$

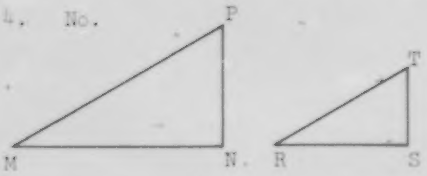
$\overline{BC} \cong \overline{DC}$ $\quad\quad$ $\angle B \cong \angle D$

$\overline{AC} \cong \overline{EC}$ $\quad\quad$ $\angle ACB \cong \angle ECD$

2. $\angle XYZ \cong \angle PQR$. No. The sides of an angle are rays and have no lengths.

3. Yes. Reasons will vary. (A formal proof is not required, but intuitive reasoning by testing several cases will show that this seems to be true.)

4. No.



The angles of these triangles are congruent respectively, but the triangles are not congruent. They are called similar.

5. $5\frac{3}{8}$" and $5\frac{5}{8}$", $\frac{1}{8}$".

6. a. $AC \approx 4\frac{11}{16}$, $AB \approx 3\frac{6}{16}$, $BC \approx 2\frac{9}{16}$

$AC = 4\frac{11}{16} \pm \frac{1}{32}$, $AB = 3\frac{6}{16} \pm \frac{1}{32}$, $BC = 2\frac{9}{16} \pm \frac{1}{32}$

b. The greatest possible error of the sum will be three times the greatest possible error of the length of any one side.

7.

	Measure	Unit
a.	3	foot
b.	17	pound
c.	24	hour
d.	16	ounce

8.

a.	$m(\angle AKB) \approx 20$	f.	$m(\angle BKE) \approx 70$
b.	$m(\angle FKE) \approx 90$	g.	$m(\angle CKD) \approx 100$
c.	$m(\angle AKC) \approx 170$	h.	$m(\angle HKD) \approx 65$
d.	$m(\angle FKG) \approx 60$	i.	$m(\angle DKB) \approx 50$
e.	$m(\angle AKD) \approx 70$	j.	$m(\angle HKC) \approx 35$

9. They are perpendicular. $\quad$ 10. They are perpendicular.

11. a. Yes $\quad$ b. Yes $\quad$ c. No $\quad$ d. Yes $\quad$ e. No

12. They are congruent.

13. They are congruent.

14. Yes

15. No

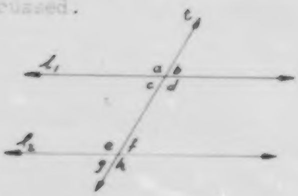
16. a. Yes, by Property II.
    b. Corresponding angles. Yes. If 2 angles are congruent, their measures are equal.
    c. Vertical angles. Yes. Vertical angles are congruent, and their measures are equal.
    d. Were drawn so as to have equal measures.
    e. The measures in the sum on the left are equal to the measures in the sum on the right.
    f. By definition of "sum."
    g. Property III.
    h. Two names for the same number as indicated in steps (e) and (g).

17. The shortest segment from a point P to a line r is the segment from P perpendicular to r.

18. The distance between two parallel lines may be described as the length of any segment contained in a line perpendicular to the two lines, and having an endpoint on each of the lines.

19. The definition of a rectangle depends on the use of a right angle which was not defined until angle measure was discussed.

20. Given: $\ell_1 \parallel \ell_2$, and transversal t.
    Prove: $\angle c \cong \angle f$
    a. $\angle c \cong \angle b$ because of Property I.
    b. $\angle b \cong \angle f$ because of Property II.



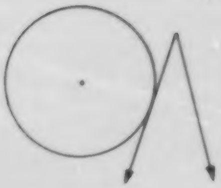
    c. $m(\angle c) = m(\angle b) = m(\angle f)$ because congruent angles have equal measures.
    d. $\angle c \cong \angle f$ because angles with equal measures are congruent.
    e. Hence if two parallel lines are cut by a transversal, a pair of alternate interior angles are congruent.

21. "Diameter" can be used to refer to the length of a line segment joining two points of a circle and containing the center of the circle. "Diameter" can also refer to the line segment itself which contains the center and has endpoints on the circle.

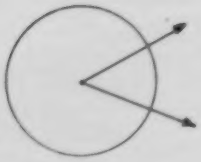
22. A diameter is a chord which passes through the center of a circle.

23. One possible answer is given for each case:

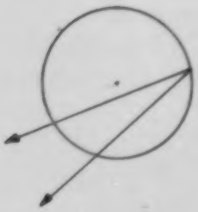
a.



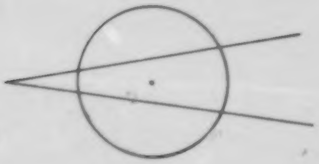
b.



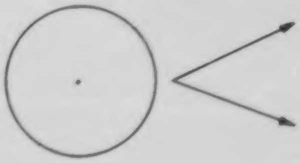
c.



d.



e.



24. The interior is the set of all points X such that PX < PR, where P is the center of the circle.

25. 90, 45, 300

26. Corresponding points may be determined in the following manner: Select any point on $\widehat{AMB}$. Draw a ray from O through that point. The ray passes through a corresponding point on $\widehat{CND}$. This may be done using points on either arc, and for any point on either arc a corresponding point on the other arc may be determined. This establishes a one-to-one correspondence between the two sets of points.

Chapter 13

PERIMETERS, AREAS, VOLUMES

Introduction

This chapter is a continuation of Chapter 12 in that we discuss the use of measurement in finding perimeters, areas, and volumes. Although there are several ways of approaching operations on numbers representing measurements, we have chosen a fairly traditional one as described in the first section.

An attempt is made to point out difficulties that students encounter in dealing with such topics as the approximate nature of measurement as it relates to perimeters, areas, and volumes, the number π, and the relationships between the various geometric figures. For example, the concept of area is approached by discussing the closed rectangular region, then relating areas of the regions of other simple closed curves to this.

A brief discussion of other units of measure relating to weight and time, along with some of the problems that students may encounter in their future studies of mathematics and science, will end this chapter.

13.1 Operations with Numbers of Measure

Binary operations on numbers have been defined in Chapter 6, but how may we define an operation on the so called "denominate" numbers? This has not really bothered us very much, but students sometimes encounter trouble both in operating with these numbers and in converting from one unit to another. Therefore, we need to consider these aspects briefly.

If we have 3 yards of ribbon and 2 yards of ribbon, how do we find the total combined length? We know how to add numbers, but "adding lengths" is something different. We could say we have two segments of 3 yards and 2 yards, respectively, laid end to end so that they have just one point in common. Then we get a segment whose measure, in yards, is 5 and whose length is 5 yards.

Let us reemphasize our terminology. Recall that in a phrase such as "3 yards is the length", we said "3 is the measure". The measure refers to the number 3. (The unit of measure is the yard.) Now we can apply arithmetic operations such as addition to these numbers called measures. If we have 3 yards of ribbon and 2 more yards of ribbon, then we have

5 yards of ribbon altogether, because the sum of their measures is 5 $(3 + 2 = 5)$.

However, we must be very careful here. For example, it makes no sense to attempt to find the sum of 35 and 17 if 35 is the degree measure of an angle and 17 is the inch measure of a line segment. We need to expand the Comparison Property of Chapter 12 which said that two continuous geometric figures or sets of the same kind may be compared as to size. Let us further agree, then, that when we operate on two numbers of measure, that they represent the same "kind of measurement", with the same unit. You have already tacitly assumed this when you did some of the exercises in Chapter 3.

In the British-American system of units there is a hodge-podge of standard units. As an example, 2 feet, 24 inches, and $\frac{2}{3}$ yards are all names for the same length, and we may use the symbol "=" to show this: 2 feet = 24 inches = $\frac{2}{3}$ yard. Also the interrelation among the units is capricious: 12 inches make a foot, 3 feet make a yard, 1760 yards make a mile.

It is important that students be able to change a measurement from one unit to another, so they must know the relationships among the units. Measurements in different units but treated as if they were in the same unit are often the basis for errors. In other words, reading the names of units as well as the number of these units, using common sense to determine which is the best unit to use for a particular problem, and being aware that operations are performed on the numbers need to be stressed with students.

As we stated earlier, most scientists and most of the non-English speaking countries of the world use the metric system of measurement. Even our units are now defined in terms of the metric system, and most rulers that children use in school today are graduated in both inches and centimeters.

Our common units were originally based on body measures and developed over a long period of time, whereas the metric system was arbitrarily made by man with no relation to his body. However, it was related to our base ten system of numeration, which allows us to handle such measurements quite easily. Let us compare base ten with the metric system.

		Length	Weight	Volume
Thousand	$1000 = 10^3$	kilometer	kilogram	kiloliter
Hundred	$100 = 10^2$	hectometer	hectogram	hectoliter
Ten	$10 = 10^1$	dekameter	dekagram	dekaliter
One	$1 = 10^0$	meter	gram	liter
Tenth	$0.1 = 10^{-1}$	decimeter	decigram	deciliter
Hundredth	$0.01 = 10^{-2}$	centimeter	centigram	centiliter
Thousandth	$0.001 = 10^{-3}$	millimeter	milligram	milliliter

In a manner much like our decimal system of numeration, each linear unit is either ten (or $\frac{1}{10}$) times as large as the adjacent unit. Thus one dekameter is the same length as 10 meters. The same relationship holds for weight and volume.

The prefixes designating positive powers of ten are adapted from the Greek and the prefixes designating negative powers of ten are adapted from the Latin. This system of units allows us to write such a phrase as: 4 kilometers 7 hectometers 2 dekameters 9 meters 8 decimeters 6 centimeters in a much simpler way: 4729.86 meters. It should be noted that the prefixes "deka" and "hecto" are seldom used. We included them for completeness.

Once students understand the prefixes and how the metric system is related to base ten, it then becomes a simple matter for them to compute with these measured quantities. For example, suppose we asked students to find the sum of 4 dekameters 6 meters 2 centimeters and 7 meters 3 decimeters 6 centimeters. This problem could be written in this form:

$$\begin{array}{r} 46.02 \text{ m.} \\ \underline{7.36 \text{ m.}} \end{array}$$

and the sum of the numbers found quite easily by the base ten addition algorithm.

Students often think that using the metric system is a "grind" or a "drag". This is usually caused by too much emphasis being placed on translating from this system to the English system and not spending enough time in looking at the metric system in its own right.

Class Exercises

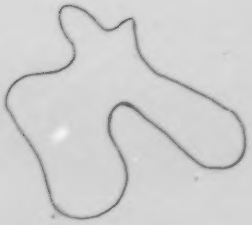
1. Divide 6 yards 2 feet 5 inches by 11.

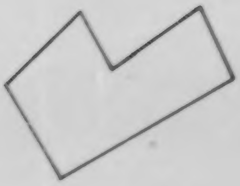
2. Divide 7 meters 6 decimeters 4 centimeters by 8.

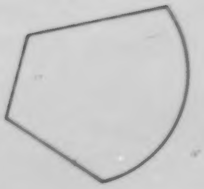
3. Which of the above problems is "easier" to do? Why?

13.2 Perimeters and Circumference

The total length of a simple closed curve is called its perimeter. In the figures below we may think of the perimeter of each figure as being the distance an ant would have to crawl along the figure in order to return to the same point from which he started.

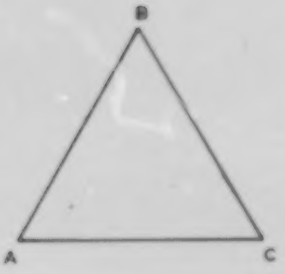






Often students think that the perimeter of a closed curve means something like "$P = 2(\ell + w)$", or "$P = 4s$", or $C = \pi d$. These are just mathematical sentences (formulas) which state precisely a recipe for dealing with the numbers used in certain geometric figures. These sentences should be the end result of the student's experiences with measuring and finding the total lengths of many closed curves, and classifying them according to some consistent pattern. Students usually have little difficulty with the concept of perimeter, even though it is subtle. Often, however, they do have difficulty with the approximate nature of measurement, "plugging" numbers into a formula which has very little meaning to them, and operating on these denominate numbers.

For instance, let us consider the perimeter of a triangle with sides of $1\frac{3}{4}$ inches each. A student has no trouble with what we mean by "perimeter" but let us explore what might happen when we ask him to find the perimeter in different ways.



The measure, in inches, of each side was given as $1\frac{3}{8}$. This immediately tells us that it was measured with $\frac{1}{8}$-inch precision and that the greatest possible error is $\frac{1''}{16}$. Therefore, we can write the length of one side as $(1\frac{3}{8} \pm \frac{1}{16})''$, and the perimeter can be expressed as $(4\frac{1}{8} \pm \frac{3}{16})''$. On the other hand, suppose we did not tell the student the measures of the sides, but asked him to measure each side to the nearest half-inch, then find the perimeter. He would report the sides as $1\frac{1}{2}$ inches each and the perimeter as approximately $4\frac{1}{2}$ inches. If the sides are measured to the nearest inch, each would be reported as 1 inch and the perimeter as approximately 3 inches. But if we ask him to lay a string as closely as possible on the segments so that these segments are all "covered", then measure the string to the nearest inch, we would expect him to say that the perimeter is approximately 4 inches. Which one is more nearly correct? As we saw in Section 2 of Chapter 12, the greatest possible error may be increased dramatically by addition or multiplication. All this example does is to point out the need again to lay careful "ground rules" for measuring and approximations.

Another common trouble spot in perimeter is computing the circumference of a circle. One of the student's first contacts with irrational numbers occurs in using π to find circumferences by the formulas $C = \pi d$ or $C = 2\pi r$. They do not realize that the symbol "π" represents an exact number, and that if we want to represent such an irrational number in decimal notation then we may do so only approximately. One state legislature even attempted, in 1897, to pass a law establishing the value of π as two rational numbers, $\frac{22}{7}$ or 3.1416.

It is interesting to note that the decimal expansion of π has been carried out to thousands of decimal places by computers, even though mathematicians have long known that it is an irrational number. The fascination of the expansion of π has intrigued people since the time of Archimedes and Pythagoras. These long computations are probably of no practical value, but the computer has helped in an examination of the distribution of the digits in the expansion of π.

Most seventh grade students have had experience in elementary school with computing an approximation to π through finding the ratio of the length of a piece of string laid around a circular object and the length of the diameter of that object. A variety of methods are available for computing approximate values of π. Often an infinite series is used to compute π. An example of one of these series is:

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3} + \frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5} + \frac{1}{3^5}\right) + \frac{1}{7}\left(\frac{1}{2^7} + \frac{1}{3^7}\right) + \ \ldots$$

A value of π correct to 55 places is given in the SMSG _Mathematics for Junior High School_, Volume I.

The important point in this discussion is that nobody has any control over the value of π: It is an irrational number. However, we may approximate π with rational numbers to any degree of accuracy we wish. We may think of it as being squeezed or bracketed between successive whole numbers, then tenths, then hundredths, and so on.

$$3 < \pi < 4$$
$$3.1 < \pi < 3.2$$
$$3.14 < \pi < 3.15$$
$$3.141 < \pi < 3.142$$

In actual practice we often use the rational numbers $\frac{22}{7}$ or 3.14 as approximations for π.

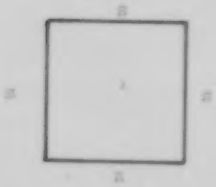
Questions usually arise with respect to how to use π in computations. If the radius of a circle is 10, then the circumference of the circle, $2\pi r$, may be written in the form 20π, which is a perfectly good number. It is the product of 20 and π. Numerically it is between 62 and 63; 62.83 correct to 2 decimal places. For many practical purposes, a satisfactory answer for the circumference of a circle is usually found by using $\frac{22}{7}$ or 3.14 as an approximation to π. We say that π is approximately equal to $\frac{22}{7}$, writing $\pi \approx \frac{22}{7}$ or $\pi \approx 3.14$. In working problems, however, we often instruct youngsters to use one of these values in their computations, and it is legitimate to say in this case: "_Let_ $\pi = \frac{22}{7}$", or "_Let_ $\pi = 3.14$". On the other hand, students in the junior high school should get lots of practice in expressing answers in terms of π as well.

Class Exercises

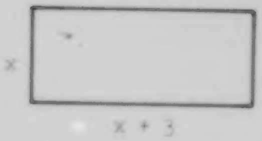
4. To five decimal places π is 3.14159 . Which is a closer approximation to this: 3.14 or $3\frac{1}{7}$?

5. State a mathematical sentence (formula) for the perimeter of each simple closed curve below:

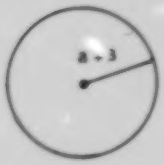
(a)



(b)



(c)



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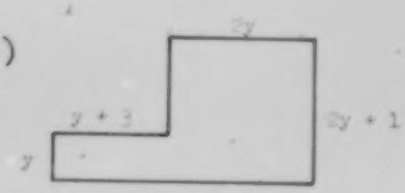


6. If a wire is strung around the equator of the earth so that it is 10 feet longer than the circumference of the earth, how far above the earth would it be? Assume that the equator is a circle and that the wire is the same distance above this circle at all points. Use $\frac{22}{7}$ for π.

13.3 Areas

In discussing perimeters, we stated that students usually had little trouble with the concept of perimeter. This is not true of the concept of area. Ask most people what the "area of a rectangle" is, and they will probably say, "It is the length times width." Again it is certainly convenient that we can find areas of closed rectangular regions by multiplying the number representing the length and the number representing the width, but this in no way conveys any idea of what area really is. Let us investigate this matter in this section.

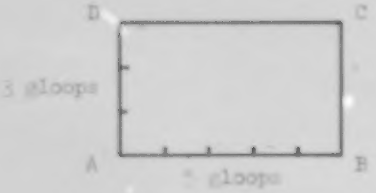
The term "area" means the measure of the closed region of a simple closed curve. In Chapter 11, closed region was defined as being the union of a simple closed curve and its interior. In choosing units of measure we agreed that our units must be of the same kind as the set of points to be measured. Therefore, in order to measure closed regions, we should choose some closed region as a unit.

We may pick any arbitrary shape for a standard unit of area. Students may approximate areas by using "units" of various shapes: circles, triangles, rectangles, hexagons, or even irregular shapes. This activity will help students understand the concept of area and perhaps convince them that it is only for convenience of communication that we adopt, as a standard unit, a closed region whose boundary is a square with each side being a standard unit of length. All measurements of area are then made by comparing against this standard unit of area.

Students are confused when they hear statements like: "Inches times inches is square inches," and "Feet times feet is square feet." Remember, in dealing with numbers of measurement, we agreed to operate on the numbers, and that operating on the names of the units has no meaning whatsoever. Even though we hear these statements often, and they are mnemonic devices, we should probably avoid them with students. We call these units square inches, square feet, or square centimeters because their boundaries are squares.

Let us agree on an item that will save us a little time and space throughout the rest of this chapter. We often hear the phrase, "area of a rectangle". We previously defined area as the measure of a closed region. A rectangle is not a closed region, even though it determines a closed region. Thus, the phrase, "area of a rectangle" is meaningless. What we really mean is the area of a closed rectangular region. However, this is quite a mouthful, and we will agree to return to our "mathematical slang" if no question of its meaning results. We use "area of a rectangle" to mean "area of the closed rectangular region".

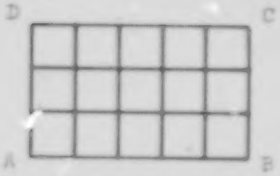
Why, then, can we find the area of a rectangle by multiplying the number representing the units of length and the number representing the units of width? Let us look at a rectangle whose length is 5 gloops and whose width is 3 gloops.



We choose a closed square region whose side has length of one gloop, and call it a square gloop.

1 square gloop

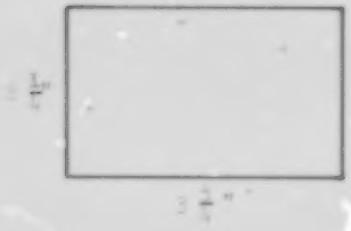
Now, how many of these congruent closed square regions are necessary to completely cover the closed rectangular region? We see that 15 are needed and we may state that the area of rectangle ABCD is 15 square gloops.



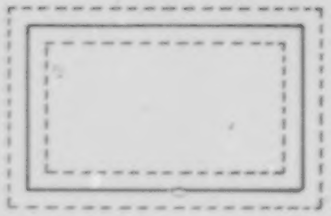
A shortcut to obtaining this area would be to consider this as a 3 by 5 array and then find the product of the numbers 3 and 5. This is what we mean when we state the mathematical sentences $A = \ell w$, or $A = lw$. The symbols A, ℓ, and w represent numbers, and the sentence $A = \ell w$ states that some number A is the product of two numbers, ℓ and w. Thus, in our figure above, we should state that the area of rectangle ABCD, in square gloops, is 15.

Again, we have idealized this situation by assigning the number 5 to the length and the number 3 to the width. Practically, in measuring, we usually encounter parts of units and either have to subdivide our unit or consider fractional parts of units. There is a large gap between the idealized situation and the practical situation that needs to be bridged carefully. A simple closed curve drawn on an overhead projector and overlayed with grids of different units helps develop this concept of area.

We should also consider greatest possible error as it relates to area. Think of physically measuring the length and width of a rectangle with a ruler whose precision is one-fourth inch, and obtaining approximate measurements of $3\frac{3}{4}$ inches and $2\frac{1}{4}$ inches.



We may write the length and width in the forms, $3\frac{3}{4} \pm \frac{1}{8}$, and $2\frac{1}{4} \pm \frac{1}{8}$. Observe that there is a largest rectangle and a smallest rectangle between which our given rectangle will lie.



This may also be shown by a table:

	Minimum Rectangle	Measured Rectangle	Maximum Rectangle
Length	$3\frac{5}{8}$ in	$3\frac{3}{4}$ in	$3\frac{7}{8}$ in
Width	$2\frac{1}{8}$ in	$2\frac{1}{4}$ in	$2\frac{3}{8}$ in
Area	$\frac{493}{64}$ = $7\frac{45}{64}$ sq in	$\frac{135}{16} = \frac{540}{64}$ = $8\frac{28}{64}$ sq in	$\frac{589}{64}$ = $9\frac{13}{64}$ sq in

From the table we see that the measured area of the rectangle lies between $7\frac{45}{64}$ sq.in. and $9\frac{13}{64}$ sq.in. The errors from the reported area of $8\frac{28}{64}$ sq.in. are $\frac{47}{64}$ sq. in. and $\frac{49}{64}$ sq. in.

The greatest possible error for this rectangle is thus $\frac{49}{64}$ sq. in., and we can indicate the precision of the calculated area by writing:

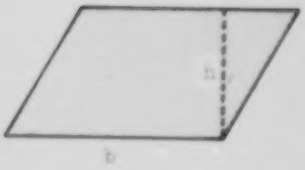
$$\text{Area} = (8\tfrac{28}{64} \pm \tfrac{49}{64}) \text{ sq. in.}$$

Usually we just find the calculated area and do not concern ourselves with the possible error; but in fields like tool design and drafting, these tolerances often are very critical.

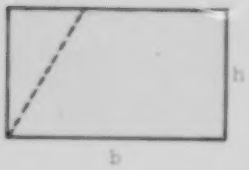
Only after the concepts of area, precision, and greatest possible error have been established should students spend time on developing the formulas for finding areas of simple closed curve regions. Let us now show one approach to these formulas. We have stated that the sentences, $A = \ell w$, or equivalently, $A = bh$, will help us find the area of a closed rectangular region. In this discussion we shall use the latter formula, where $\underline{b}$ is the measure of the length and $\underline{h}$ is the measure of the width of a rectangle. An attempt will be made to relate the formulas of parallelograms, triangles, trapezoids, and circles to this.

If we are given a model of a closed region representing a parallelogram, then this model may be cut and reassembled in such a way so as to make it look like a closed rectangular region. See the figures below.

(a)

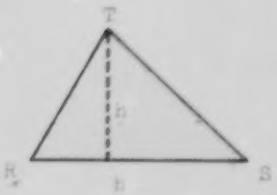


(b)

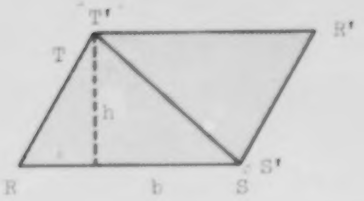


It may be proved that the figure on the right is indeed a rectangle whose area is given by the product bh. Our Subdivision Property, which tells us the two areas are the same, now allows us to state that the formula for the area of the parallelogram is also given by the formula $A = bh$.

Areas of triangles may now be related to areas of parallelograms. Think of a model of any closed triangular region such as is pictured below. The height of a triangle is defined as being the length of the perpendicular from the vertex T to the base $\overleftrightarrow{RS}$.

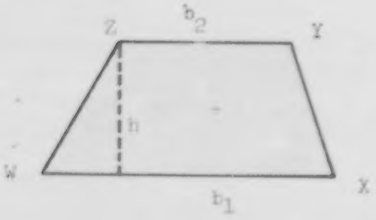


Now consider another model, $\triangle R'S'T'$, congruent to $\triangle RST$, and place it in the position shown below.

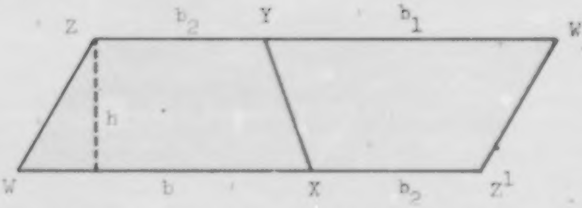


It can be proved that figure RSR'T is a parallelogram, but we will accept this as being true. Observe that the area of parallelogram RSR'T, $A = bh$, is twice as large as the area of the triangle. Therefore, we may state the formula for the area of a closed triangular region as $A = \frac{bh}{2}$.

Moving on to the area of the closed region of a trapezoid, we shall need to add a little notation. A trapezoid has two sides parallel, and both are often called bases. Let us call the bases b_1 and b_2, as in the following model.



If another model congruent to WXYZ is made and placed as in the diagram below, it is possible again to prove that the resulting figure is a parallelogram. We will accept this as true, also.

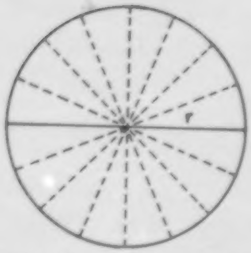


The area of this parallelogram WZ'W'Z may be found as the product of the height and base. As the length of the base may be expressed as $(b_1 + b_2)$, then the formula for the area of the larger figure may be expressed as: $A = (b_1 + b_2)h$. However, the two trapezoids were congruent, and our area is again twice as large as we wish. Therefore, the formula to help us find the area of our original trapezoid may be stated as:

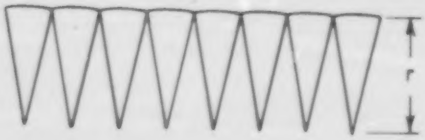
$$A = \frac{(b_1 + b_2)h}{2}.$$

The last formula that we will develop here is the one for the method of computing the area of a closed circular region in terms of the radius of a circle. There are several possible approaches, and many are discussed in SMSG Mathematics for Junior High School, Volume I. We have said that we would relate our formulas to the formula for the area of a rectangle. Let us pursue this train of thought by trying to transform a model of a closed circular region into a model of a parallelogram, then applying the formula, $A = bh$.

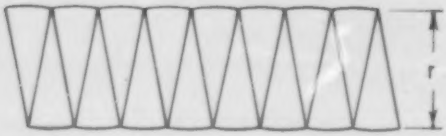
Let us imagine drawing a large circle with several radii, as shown below, so that all the central angles are congruent. For convenience we chose 16 central angles. Note also that two semicircles are formed.



Now imagine cutting around the circle, then cutting it in two, then cutting along the dotted lines. Eight of these angular portions should look something like this when carefully laid out:



If both portions are cut in this manner and fitted together, then we would have something like the figure below.



The upper and lower boundaries of the completed pattern have a scalloped appearance. If, in the same manner, we cut the circular region into smaller and smaller slices, it would seem that the boundaries would approach the appearance of the following figure:

But this is a rectangle and the area may be found by the sentence $A = bh$! All we have to do is determine the measures that correspond to b and h. Do you see that the measure of the base will be approximately one-half the measure of the circumference? In the last section, the relation of the circumference to the diameter and the radius was stated as $C = \pi d$ or $C = 2\pi r$. One half of the circumference then would be just πr. Now, if we can state the height, h, in terms of the radius, we will have our problem solved. Notice, however, the measure of the height is the same as the measure of the radius of our original circle. Therefore, in the formula $A = bh$, we may substitute "πr" for "b" and "r" for "h", obtaining:

$$A = \pi r \cdot r$$

or

$$A = \pi r^2 .$$

This is the well-known formula for finding the area of a circle. Remember, this has been strictly an intuitive approach that seems to suggest the formula for the area of a circle. Nowhere have we proved that this is true. We shall leave the proof for later courses in mathematics.

We have developed a few of the more familiar formulas for areas. Many other simple closed curved regions may be subdivided into these common figures so that their areas may be computed. This is not the only approach and these formulas are not the only ones; there are many ways to present these ideas. We have taken a strictly intuitive approach, but students will encounter more sophisticated methods as they continue their mathematics education.

Class Exercises

7. In a rectangle, does the length always have to be longer than the width? Explain.

8. How would you justify the statement, $A = s^2$, as the area formula for a closed square region?

9. If a farmer has 100 feet of fencing, what is the approximate area of the largest garden he may enclose with this fence?

13.4 Measurement of Solids

The concept of volumes of solid regions is a bit more difficult than that of areas of plane regions primarily because students have trouble visualizing solid regions when the diagrams of these are always in a plane. As was suggested in Chapters 10 and 11, sketches and models of solid figures made by the students will help them understand three dimensional space better. The use of 1-inch cubical blocks to "fill" a model of a solid, models of a cubic foot, a cubic yard, and so on, also enable students to picture the volume concepts a little clearer.

The discussion of the previous section relative to area also applies to volume, and we will not spend much time repeating many of these topics. In other words, we should proceed with students in a manner similar to the way in which linear, angular, and area measurements were developed. Let us briefly mention these ideas again.

Recall that we have said that, theoretically, a continuous quantity may have an exact measure, but that practically it never does. For example, we are talking theoretically when we say a segment has a length. We are talking practically when we say its length is a particular measure correct to a certain number of places. We have also said that the set to be measured must be measured by some unit of the same kind: a unit segment to measure segments, a unit angle to measure angles, and a unit closed region to measure closed regions. Similarly, we need to choose some unit solid to measure solids.

Let us pause for a moment and consider our terminology. In Chapter 11, we did not define right prisms because the ideas of congruence and angle measures had not been discussed. A right prism is a prism in which the lateral edges are perpendicular to the bases. All lateral faces of a right prism are rectangular regions. A right rectangular prism is a prism whose opposite faces are congruent rectangular regions. The term right rectangular solid will refer to the set of points consisting of a right rectangular prism and its interior. The volume of a particular solid is the number assigned to the measure of the space it occupies. We will usually speak of the volume of a right rectangular prism: by this we really mean the volume of the corresponding solid. In other words, the volume is associated with the solid and not with the surface which bounds the solid.

A cube may be defined as a right rectangular prism whose edges are all congruent. A cubical solid is the usual choice for a unit of volume, and through discussion with students, they soon realize that this is the preferred unit.

If we wished to find the measure of the surface of a solid, this would be called the surface area. Surface area will not be discussed here except to say that the areas of the faces of a solid figure may be found as in the last section, then the sum of these areas would be the surface area of our solid. Students can often be helped in determining surface areas by "opening up" the paper models of the solids they have constructed.

Two other aspects that we discussed in detail previously and that should be related to volume are the development of the standard formulas and the greatest possible error. Let us consider the formulas first. The volume of a rectangular solid is measured by the number $\ell \times w \times h$, where ℓ, w, and h represent the measures of length, width, and height in the same units. This may be expressed by the familiar formula:

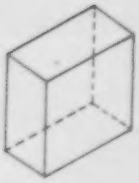
$$V = \ell wh.$$

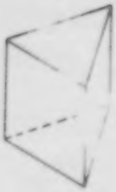
Since the measure of the area of the base is equal to "$\ell \times w$", we frequently say that the volume of a right rectangular prism is the product of the area of its base by its height. Letting B stand for the measure of the area of the base, this becomes: $V = Bh$.

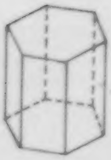
The importance of developing the concept of volume before the formulas cannot be stressed too much. Students do not really need formulas if they understand the concept; they can always develop their own recipes if volume is understood. The formulas state in concise mathematical sentences how to deal with the numbers involved.

Just as the formulas for areas of closed regions were all related to the area of a rectangle, the formulas for certain other volumes could all be related to the volume of a right rectangular solid. We may first consider right prisms with different shaped bases and see that the volume is equal to the area of its base times its height:

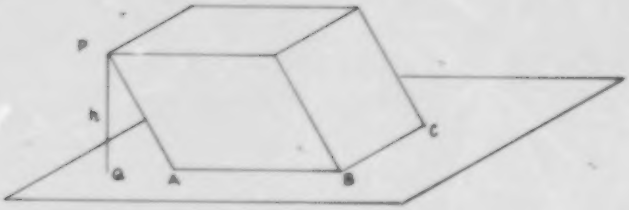
$$V = Bh.$$



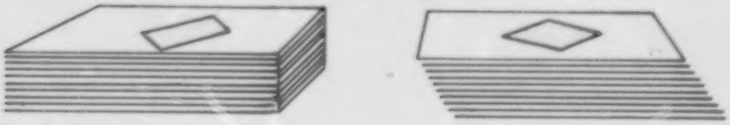




An oblique prism such as pictured in the drawing below

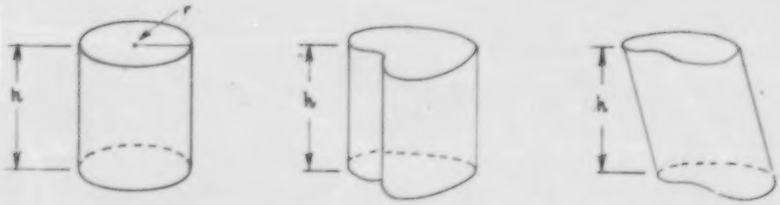


may be thought of as a deck of cards which has been pushed into an oblique position but still having the same volume as the corresponding right prism. It differs from a right prism in that its lateral edges, while still congruent, are not perpendicular to the bases. Also its lateral faces are not necessarily rectangular.



The only word of caution needed here is that we refer to the height of this oblique prism as the length of PQ, not the length of a lateral edge.

The same approach with slight modification can be made to apply to volumes of cylinders.



In each case shown above, the volume is given by the product of the area of the base and the altitude. The right circular cylinder on the left has volume given by $V = \pi r^2 h$, where πr^2 gives the area of the base.

We may state in general that for any prism or cylinder, right or oblique:

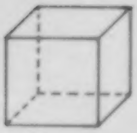
$$V = Bh .$$

Formulas for volumes of solid regions bounded by pyramids, cones, and spheres are more difficult to justify in the way that we have been proceeding, and these are not often developed for seventh grade youngsters. We may, however, make the following relationship between pyramids and prisms, as well as cones and cylinders, plausible by using hollow models and water or sand to establish their relative volumes. By this method we can show that

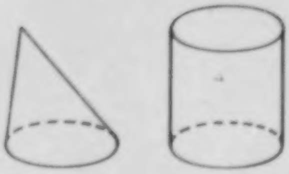
$$\text{for any pyramid or cone:} \quad V = \frac{1}{3} Bh.$$

Study the figures below.

(a)



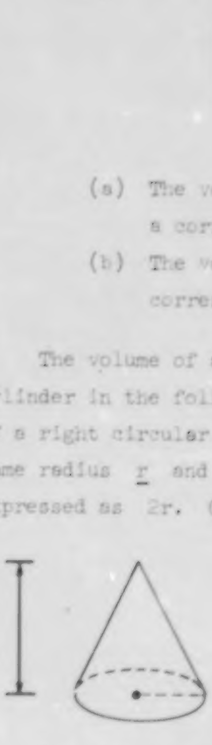
(b)

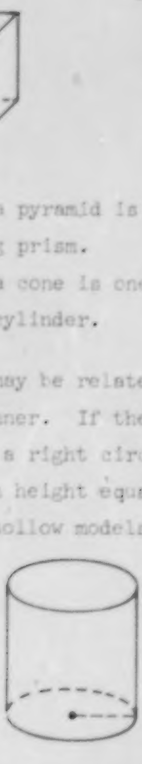


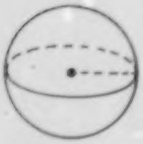
(a) The volume of a pyramid is one-third the volume of a corresponding prism.

(b) The volume of a cone is one-third the volume of a corresponding cylinder.

The volume of a sphere may be related to the volumes of a cone and a cylinder in the following manner. If the radius of a sphere is r, think of a right circular cone and a right circular cylinder each with the same radius r and each with height equal to the diameter of the sphere, expressed as 2r. Consider hollow models of each as in the drawing below.







Now, if we asked students to perform the following experiment, certain results would seem to be indicated. If the cone is filled with sand and this sand is poured into the cylinder, we know from the previous experiments, the cylinder will be about one-third full. If the sphere is also filled with sand and then emptied into the cylinder which is already one-third full with sand from the cone, the cylinder will appear to be completely full. Several trials will convince students that the volume of the sphere seems to be two-thirds that of the corresponding cylinder and twice that of the corresponding cone. Since the radius of the base of the cylinder is r and its height is 2r, the volume Bh is

$$V = (\pi r^2) \times (2r) .$$

Therefore, the volume of the sphere is

$$V = \frac{2}{3} \times (\pi r^2) \times (2r)$$

or

$$V = \frac{4}{3} \pi r^3 \ .$$

From this experiment, we are fairly sure that $V = \frac{4}{3} \pi r^3$, but remember that we still have not proved it. A physical measurement can not prove a mathematical idea, only suggest it and support it. We will leave the formal proof of this for a more sophisticated course in mathematics.

The other aspect we mentioned earlier regarding greatest possible error is the last topic in this section to be discussed. Recall that we observed that the multiplication of two numbers used in measurement quickly increased the greatest possible error. The involvement of a third number in computing volumes quite radically increases this again. A large amount of classroom time probably should not be spent on this topic, and the use of an overhead projector will help accelerate the presentation and understanding of greatest possible error as related to volumes. For example, consider a right rectangular prism measured with one-half inch precision with the following dimensions: $\ell = 10\frac{1}{2} \pm \frac{1}{4}$, $w = 3\frac{1}{2} \pm \frac{1}{4}$, and $h = 5 \pm \frac{1}{4}$. A table similar to the one used for rectangles in the preceding section of this chapter could be drawn beforehand on the overhead projector and completed by the class. This method would show the development of the problem and is quite effective with students. We will not do the mechanics of the computation, but the greatest possible error in volume here is 27.89 cubic inches. This seems large for the measurements originally made to the nearest half-inch, but illustrates the rapid increase possible in such calculations.

Class Exercises

10. Suppose ℓ and w of a right rectangular prism are each doubled and the lateral edge left unchanged. What is the effect on the volume?

11. What is the effect on the volume when each of ℓ, w, and h of a rectangular prism is doubled?

12. The sides of the square base of a pyramid are doubled and the height is halved. How is the volume affected?

13. If a truck is called a 5 ton truck when its capacity is 5 cubic yards, then what is a truck called which has a body 6 feet wide by 9 feet long by 5 feet high?

14. Compute the greatest possible error in the example given in the last paragraph, if the measurements are made with one-quarter inch precision, i.e., $\ell = 10\frac{2}{4} \pm \frac{1}{8}$, $w = 3\frac{2}{4} \pm \frac{1}{8}$, and $h \doteq 5 \pm \frac{1}{8}$.

13.5 Conclusion

Several topics about geometry, both metric and nonmetric, have not been mentioned in these last few chapters, but not because they are unimportant. We should not be left with the impression that only lengths, angles, areas, and volumes are measured. Time, weight, and mass, as well as other quantities, could have been presented here, too: but a discussion of one topic like area was considered in depth rather than lightly covering many ideas. Many definitions were not stated, either, but may be found in SMSG Mathematics for Junior High School, Volume I. It is hoped that the presentation here will furnish you with methods of introducing these other topics to students. Much of this material on measurement has always been included even in the most traditional textbooks, but students often have not really understood the concepts involved.

As you have probably observed, measurement is the vehicle by which mathematics is related to the physical world, it is the language of science. Interesting examples of how mathematics may be introduced through measurement and scientific experiments may be found in the SMSG publication, Mathematics Through Science. Students should find in this book some different approaches to the development of some of their mathematical concepts.

Scientific and engineering problems are requiring more and more precise measurements and measuring devices, and new units of measure are invented to meet these needs. For example, an angstrom is a unit of length which is one hundred millionth of a centimeter, and a micro-second is a unit of time which is a millionth of a second. These units are very small. On the other hand, astronomers also need very large units such as the light year which is the distance light travels in one year at approximately 186,000 miles per second.

Students should remember that measurement is always approximate, and answers are expressed to the nearest unit, whatever unit is being used.

Also, a decision must be made by the student as to which unit is the most appropriate for any particular problem. Seventh grade youngsters should begin to have some exposure to a few of the unfamiliar units of measure as well as the relationships between these and the more common ones.

Answers to Class Exercises

1. 1 ft., $10\frac{1}{11}$ in.

2. 0.905 m.

3. The second problem is easier because we can use the standard base ten division algorithm immediately.

4. $3\frac{1}{7}$ is a closer approximation to π than 3.14 .

5. (a) $P = 4s$ (b) $P = 4x + 6$ (c) $P = 2\pi(s + 3)$ (d) $P = 10y + 8$

6. The wire would be approximately $\frac{35}{22}$ or $1\frac{13}{22}$ feet above the earth at all points. The circumference of the earth can be represented by $C = 2\pi r$. If the circumference is increased by 10 ft. then the radius is increased by x ft. and we have

   $$C + 10 = 2\pi(r + x) .$$

   But $C + 10 = 2\pi r + 2\pi x$

   thus $10 = 2\pi x$ and $x = \frac{10}{2}\pi$.

   It is interesting to note that the problem can be solved without ever knowing the radius or circumference of the earth.

7. In everyday usage we think of the length as being longer than the width, but it makes no difference which is the length and which is the width because this may be interpreted as an application of the commutative property of multiplication.

8. Using the formula for the area of a rectangle: $A = \ell w$, and realizing that a square is a special kind of rectangle, allows us to substitute $\underline{s}$ for both ℓ and w.

9. It is a closed circular region with an area of approximately 795 square ft.
10. The volume is 4 times as great.
11. The volume is 8 times as great.
12. The volume is twice as great.
13. It is a 10-ton truck.
14. $13\frac{329}{512}$ or 13.64 cu. inches .

Chapter Exercises

1. The measures of the sides of a triangle in inch units are 17, 15, and 13.
   (a) What would be the measures of the sides if measured to the nearest foot?
   (b) What is the measure of the perimeter in inches? In feet?
   (c) How do you explain what seems to be an inconsistency?

2. Which plane region has the greater area - a region bounded by a square with a side whose length is 3 inches or a region bounded by an equilateral triangle with a side whose length is 4 inches?

3. Here is a problem which your students might do: Take an ordinary half dollar.
   (a) Trace an outline of it on a graph paper grid with unit $\frac{1}{10}$ inch. Estimate the area by using the grid.
   (b) Use thread to represent the circumference and radius, measure them on the graph scale, and use them to compute the area.
   (c) Compare the two results.

4. (a) A child measures a rectangular prism with a ruler whose unit is an inch and obtains these measurements: length, 5 inches; width, 3 inches; height, 6 inches. What is the volume?
   (b) The same prism is measured with a ruler whose unit is 0.1 inch. The length is now reported as 5.2, the width as 3.4, and the height as 6.3 inches. What is the volume?
   (c) How do you explain the large discrepancy in the answers to (a) and (b) ?

5. A cone has height .12 feet and base a circle of area 6 square feet. What is the height of a cylinder whose base and volume are equal to that of the cone?

6. Find the volume of a ballbearing whose radius is 0.1 inch.

7. The radius of an unopened tin can is 2 inches and the height is 3 inches.
   (a) What is the circumference of the base?
   (b) What is the volume of the can?
   (c) What is the total surface area of the can?

8. A rectangular prism is measured to be 10" by 8" by 6" with 1-inch precision.

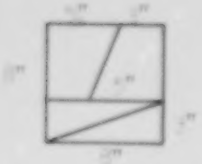
(a) What is the smallest possible measure of the true length? Width? Height?

(b) What is the largest possible measure of the true length? Width? Height?

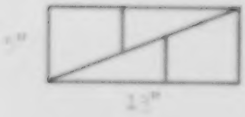
(c) What is the smallest possible measure of the true volume?

(d) What is the largest possible measure of the true volume?

9. (a) Consider a model of a square region with a side of 8 inches and cut along the lines as in the diagram below. What was the area of this square?



(b) The pieces cut from the square may be placed so as to form a rectangle similar to the following. What is the area of this rectangle?



Note: Students enjoy this problem and invent several theories about why this paradox seems to happen.

10. If the radius of a circle is doubled, what is the effect on the circumference? What is the effect on the area?

Chapter 14

DESCRIPTIVE STATISTICS AND PROBABILITY

Introduction

The gathering, summarizing, and presenting of data is an important and common activity today. Information is presented daily in various media by tables, charts, and graphs. A variety of descriptive terms are used to summarize large quantities of data. While most people are not directly concerned with the preparation of such data, every educated person should have some ability to correctly interpret statistical data. For this reason descriptive statistics is introduced at the junior high level. The main points discussed here are graphing of data, and measures of central tendency and dispersion. In each case solving problems of this nature gives students an understanding and an ability to interpret information more clearly. Having made several broken line graphs and bar charts, they find little difficulty in reading and interpreting such graphs.

The gathering of data may range from simple reference work such as looking up previously recorded information, to the more sophisticated random sampling procedures used in various types of quality control. Although we will not be concerned here with the problems of sampling, students are quick to see some of the flaws inherent in different sampling methods and enjoy discussing this topic. Information for such work is easily obtained. Student heights, weights, distance from home, number of brothers and sisters, ages, are all easily obtained and lend themselves to statistical treatment.

14-1 Graphing

Having obtained a set of data by some means, we are usually confronted with the task of organizing and preparing it for presentation. Often, sets of data may be presented in table form as the example below. However, it is usually difficult to abstract information from tables. Graphs are generally clearer, easier to read, and often show relationships not readily apparent in a table.

Population Facts About the United States

Census Years	Population in Millions	Increase in Millions	Percent of Increase
1790	3.9		
1800	5.3	1.4	35.1
1810	7.2	1.9	36.4
1820	9.6	2.4	33.1
1830	12.9	3.3	33.5
1840	17.1	4.2	32.7
1850	23.3	6.1	35.9
1860	31.4	8.2	35.6
1870	39.8	8.4	26.6
1880	50.2	10.4	26.0
1890	62.9	12.7	25.5
1900	76.0	13.1	20.7
1910	92.0	16.0	21.0
1920	105.7	13.7	14.9
1930	122.8	17.1	16.1
1940	131.7	8.9	7.2
1950	150.7	19.0	14.5

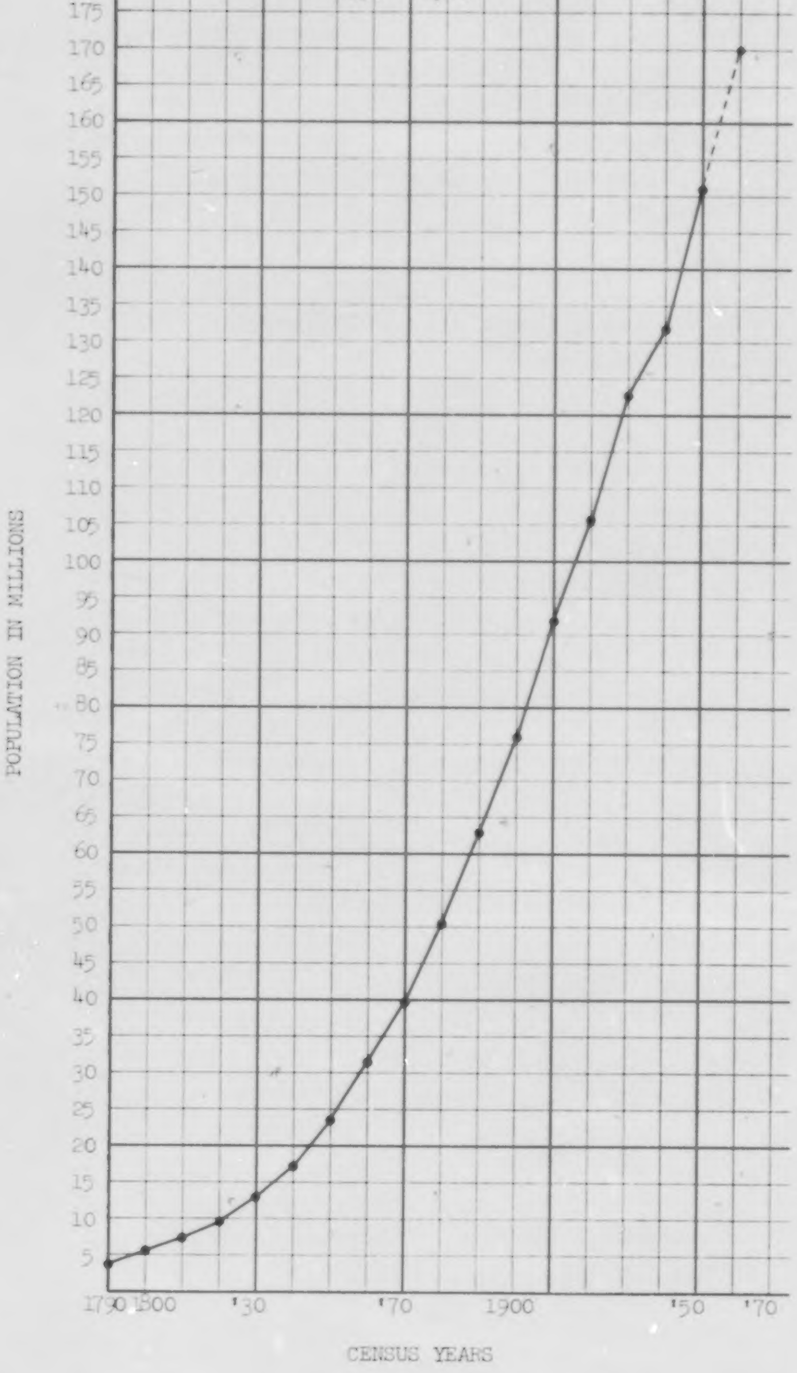
The broken-line graph is a common way of picturing data. Such a graph is made by first locating points on graph paper and then connecting them consecutively with line segments. The graph below shows the data in the table given previously. Here it is easy to see the changing rate of population increase, the decrease in rate during the 1930's, the population in the years labeled, as well as an approximation to the population at any given time.

Students generally need help in the preliminary work which must be done before any actual graphing takes place. One of the biggest problems in constructing broken-line graphs is deciding upon the scale. How much each unit space should represent so that the graph is of the appropriate size must be decided before any points are put on the paper. Some students will even need step by step instructions as to how to decide on the scale to be used. Such directions as, "count the number of spaces available, divide into the largest quantity to be shown on the graph paper, and round off to the next larger unit," may be necessary.

Bar graphs are another way of representing data graphically and are also relatively simple to construct. The same problem of scaling occurs as in drawing a broken-line graph. Once they have mastered the basic techniques, students mainly need practice in making neat, clearly labeled graphs which display the desired information.

POPULATION OF THE UNITED STATES

1790 - 1960



Circle graphs are still a third type of graph with which students must be familiar. Their preparation requires the use of a protractor and some calculation as to the size of angles needed in a particular graph. Ratio and proportion or percent are usually needed. Thus to prepare a circle graph of the data presented in the table below we need to determine the size of each angle.

Fruit Preference for Lunch

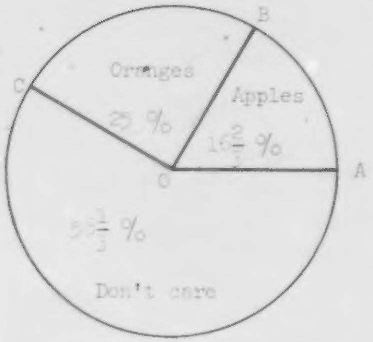
Apples	8
Oranges	12
Don't care	28
Total	48

To do so we need either the percent or fractional part of the total each observation represents. Both are given below.

	Number	Fractional Part	Percent	Degrees
Apples	8	1/6	16 2/3	60
Oranges	12	1/4	25	90
Don't care	28	7/12	58 1/3	210
Total	48	1	100	360

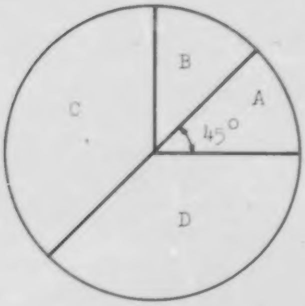
In either case we see that an angle of 60° will represent the 8 votes for apples, since $\frac{1}{6}$ of 360 is 60 and $16\frac{2}{3}$ percent of 360 is 60. Of course all problems will not give such exact results but rounding off to the nearest degree will usually be as accurate as necessary for most graphs.

Fruit Preference for Lunch



Class Exercises

Use the figure below to answer questions 1-3 .



1. What percent of the circular region is region A ?

2. How many degrees should be in the central angle if region C is to be $37\frac{1}{2}$ percent of the total area?

3. If D is the same size as C how many degrees are in the central angle of region B ?

4. Make a broken-line graph to show a possible trend in the 12 successive test scores given: 72, 80, 77, 95, 84, 1, 98, 75, 80, 100, 67, 77.

5. Show the data in exercise 4 by means of a bar graph.

14.2 Summarizing Data

Although information presented in graphical form is often easy to understand, we may want to know more about the data. Two questions which generally arise are, "What is an average or typical figure?" and, "How much do the observations differ from this average?" In the first question we are looking for a single number which can be used to represent all the data. In the second question we are concerned with how the various observations are distributed about this average. Some sets of observations are spread over a wide range while some may be very close together. The terms used to answer the first question are measures of central tendency. The terms used to answer the second are measures of dispersion.

Mathematicians have three technical terms used to measure central tendency. They are mean, median, and mode. Each of the three gives a number which in some sense may serve to represent all the data. Unfortunately each is associated with the word "average".

The mean or arithmetic mean is what most people generally think of when they use the word "average". The mean of a group of numerical observations is calculated by adding all the observations and dividing that sum by the number of observations. Consider a small company of nine employees with salaries as shown below. Adding the salaries and dividing by nine gives a mean salary of $14,000 .

$ 45,000	(President)
35,000	(Son-in-law)
10,000	(Vice-president)
9,000	(Custodian)
7,000	(Treasurer)
6,000	(Designer)
5,000	(Salesman)
5,000	(Salesman)
4,000	(Production)

Although the mean is frequently used, at times it may be misleading. In attempting to recruit a new employee to the company, it was pointed out that the "average" salary in the company was $14,000 . It is true that this is the mean salary, and the average of $14,000 does in a way represent all the data. On the other hand it seems misleading and we are not comfortable with it since seven of the nine salaries are less than this average salary. This is one characteristic of the mean. It is sensitive to observations such as the president's salary, which differ markedly from the others.

Another type of average, not affected by a few observations which deviate markedly from the others, is the median.

The median is defined to be the middle number when data is ordered with respect to size. If there is no middle number, as is the case when the total set contains an even number of elements, then the median is the arithmetic mean of the two middle numbers. Thus, in the example above, $7,000 is the median salary. This seems to be a more significant figure than the mean in this case, since now half the salaries are higher (or equal), and half the salaries are lower (or equal). You recognize the median as the 50th percentile, a term used in reporting test data. Notice that the median would remain unchanged even if the President's salary were doubled, while the mean would be changed sharply to $19,000. We should not fault the mean for being affected by individual observations: it may be that this is the exact point we wish to emphasize.

Still another measure of central tendency is the mode. The mode is defined to be the number which occurs most often in a group of observations. Using our previous example, we see the mode to be $5,000 . Occasionally, a set of data will have more than one mode.

These three measures, mean, median, and mode, are all used at times to describe central tendency. Any time reference is made to an average we must understadd what measure is being used. Either careless or deliberate misuse of these terms can lead to erroneous conclusions. Thus the saying, "Figures don't lie, but liars figure."

It is important for students to realize that very different sets of data may have the same measure of central tendency. Consider a second company of nine employees. Salaries for this company, Company B, are listed with salaries of the previous company, Company A, for comparison.

Company A	Company B
$ 45,000	19,000
35,000	18,000
10,000	17,000
9,000	16,000
7,000	14,000
6,000	12,000
5,000	11,000
5,000	10,000
4,000	9,000

Examining the salaries as displayed in tabulated form shows a very different salary structure: for instance the lowest salary in Company B is greater than the five lowest in Company A. On the other hand, both companies have the same mean salary, $14,000 . An important difference between these two situations is the difference between the highest and lowest salary in each case. In Company A this difference is $41,000 while in Company B it is only $10,000 . This number, the difference between the largest and smallest number in a set of observations, is called the range. We see that the smaller the range the closer the individual members of the set are to the measures of central tendency: that is, the closer they "cluster" about the mean. The range then gives us some indication of how the data is distributed about the mean. It is a measure of dispersion.

Another measure of dispersion is the average deviation from the mean. The average deviation is computed by finding the difference between each number and the mean, and then finding the mean of these differences. This gives us "on the average" how much each individual observation deviates from the mean.

Let us refer again to our companies, each with a mean of $14,000, and compute the average deviations in each case.

Company A (mean $14,000)		Company B (mean $14,000)	
Salary	Deviation from mean	Salary	Deviation from mean
45,000	31,000	19,000	5,000
35,000	21,000	18,000	4,000
10,000	4,000	17,000	3,000
9,000	5,000	16,000	2,000
7,000	7,000	14,000	0
6,000	8,000	12,000	2,000
5,000	9,000	11,000	3,000
5,000	9,000	10,000	4,000
4,000	10,000	9,000	5,000
	104,000		28,000

Average Deviation

$11,544 $\left(\frac{104,000}{9}\right)$

Average Deviation

$3,111 $\left(\frac{28,000}{9}\right)$

Here again the relative sizes of the average deviations gives us information on the scatter of the data about the mean. Although other measures of central tendency are more commonly used, the average deviation is easy to compute and does give us an indication of dispersion.

The range has the disadvantage that it is affected by individual observations, and thus may not always give an accurate picture of the distribution. The average deviation is less influenced by any one observation and thus gives a better indication of the scatter of the data.

You are familiar with other measures of dispersion such as standard deviations and variance, but these are much more difficult to compute and their interpretation requires much more time than is generally available in grade seven.

Class Exercises

6. Find the mean, median and mode of the following observations: (4, 5, 5, 5, 5, 6, 8, 8, 10, 10, 11) .

7. What is the range of the above data?

8. Find the average deviation from the mean for the distribution in Exercise 6.

14.3 Probability

The study of probability and its applications is an important part of many disciplines. Relatively simple ideas which can be expressed in terms of coins, cards, dice, and marbles in bags, have developed into a powerful tool used in a wide variety of areas. The methods of statistical inference developed from the ideas of probability are used in making decisions in such diverse areas as medical research, quality control, and insurance. An understanding of some of the key ideas of probability should be part of every junior high school student's education. These ideas are relatively simple to grasp and can be used to answer a variety of questions about chance events.

When we talk about the probability of some event occuring we are asking the question, "How many times can we expect an event to occur in a given number of trials?" In the simple example of a coin we see that when flipped in the air it can land two ways, either heads or tails. It seems reasonable that one outcome is just as likely to occur as the other and we would expect to obtain about twenty-five heads and twenty-five tails in fifty trials. We would say that the ratio of the number of heads to the number of trials is $1 : 2$. Since this means that about half the time we would get a head, we say that the probability of getting a head is $\frac{1}{2}$. The same reasoning leads us to expect a given number, say a 3, about one out of six times when rolling an ordinary die. We would expect the ratio of the number of threes to the number of rolls to be $1 : 6$. Again we would say the probability of getting a three is $\frac{1}{6}$. Notice that in these cases only one of the possible outcomes can occur at a time and each appears equally likely.

This idea leads us to one of the basic notions of probability. If all the possible outcomes of an experiment are equally likely, then we may express the probability that an event E will occur as

$$P(E) = \frac{t}{s}$$

where t is the number of possible outcomes in which event E occurs, and s is the total number of possible outcomes.

Thus the probability of a head showing on a single toss of a coin is

$$P(H) = \frac{1}{2}$$

since, of the 2 possible, equally likely outcomes (H and T), only 1 (H) is a success.

The probability of a 6 showing on a single roll of a die is

$$P(6) = \frac{1}{6}$$

since, of the 6 possible, equally likely outcomes (1, 2, 3, 4, 5 and 6), only one (6) is a success.

What can we say about the number $\frac{t}{s}$ in the probability formula $P(E) = \frac{t}{s}$? If every possible outcome is considered a success, then $t = s$, $\frac{t}{s} = 1$, and the probability of success is 1. If no outcome is considered a success, $t = 0$, $\frac{t}{s} = 0$, and the probability of success is 0.

If an event A is certain to occur, then $P(A) = 1$.

If an event B cannot occur, then $P(B) = 0$.

Further, we may write

$$0 \leq P(E) \leq 1 \quad .$$

As the probability changes from 0 toward 1, we become more and more certain of success.

Example: What is the probability of drawing the four of hearts from an ordinary deck of 52 playing cards?

Solution: Since out of 52 possible outcomes a success can occur in only one way, the probability is $\frac{1}{52}$. We assume each card has an equal chance of being drawn.

Example: What is the probability of drawing an ace from the same deck of 52 playing cards?

Solution: Here we may draw any one of the four aces so that a success may occur four ways out of the 52 possible outcomes. Thus the probability of an ace is $P(\text{ace}) = \frac{4}{52} = \frac{1}{13}$.

Class Exercises

9. What is the probability of getting an even number in rolling an ordinary die with six faces numbered 1, 2, 3, 4, 5, 6 ?

10. What is the probability of getting a prime number in rolling the die in Problem 9?

11. What is the probability of drawing a five from an ordinary deck of 52 playing cards?

12. What is the probability of drawing a red five from an ordinary deck of 52 cards?

Since the probability of event A is given by $P(A) = \frac{t}{s}$ then the probability of A not occurring will be given by

$$P(\text{not } A) = \frac{s - t}{s}\ .$$

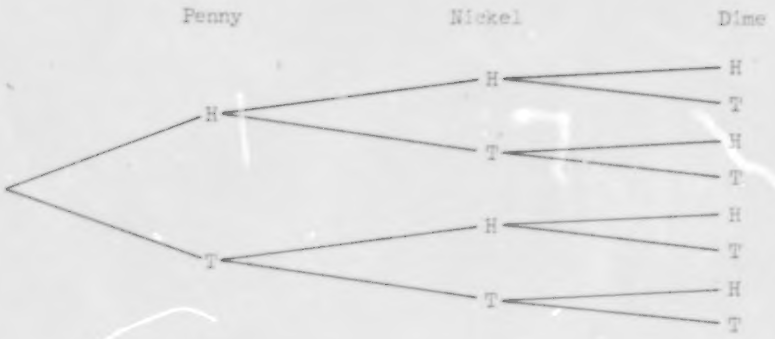
(This is so, because if A can occur in t ways, then it will fail to occur in $s - t$ ways.) But changing the form of this fraction gives the following:

$$\begin{aligned} P(\text{not } A) &= \frac{s - t}{s} \\ &= \frac{s}{s} - \frac{t}{s} \\ &= 1 - \frac{t}{s} \\ P(\text{not } A) &= 1 - P(A)\ . \end{aligned}$$

Therefore the probability of an event not occurring is 1 minus the probability of the event occurring. This seems necessary since we want the sum of the probabilities for any particular situation to add to 1.

$$P(A) + P(\text{not } A) = 1$$

To answer many questions of probability we need a method of determining all possible outcomes of certain types of events. One way of listing the outcomes is illustrated below. Suppose we wish to enumerate the possible outcomes in flipping a penny, nickel, and dime. The tree diagram below shows all possible arrangements for the three coins.



From this tree we see there are eight possible outcomes, listed below.

POSSIBLE OUTCOMES

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Since each of these possible outcomes is equally likely, we assign to each the probability $\frac{1}{8}$. The sum of the probabilities for all possible outcomes in this situation, as in all cases, is 1. We are now in a position to answer questions such as the following, "What is the probability of getting 2 heads and one tail when three coins are flipped?" Referring to the table we see that 2 heads and one tail can occur three ways out of the eight, so that the probability is $\frac{3}{8}$.

Class Exercises

Use the table developed above to answer the following:

13. What is the probability of getting at least two heads?

14. What is the probability of all three coins being the same?

14.4 Probability of A or B

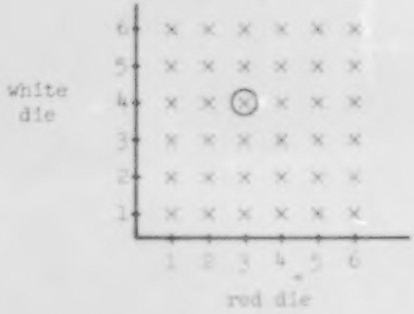
Our previous discussion was limited to single events. Other situations arise when we want to know the probability that one of two or more events occurs. Let us consider the possible outcomes if we roll two dice and record the numbers showing. We could use a tree to list all possible outcomes but another way would be to think of the two dice as being different colors, say red and white. Then we see that we could get a red 1, with any face of the white, i.e., R1-W1, R1-W2, R1-W3, R1-W4, R1-W5, R1-W6. The same possibilities exist for a red 2, a red 3, and so forth. This leads us to the table below.

Possible Outcomes with Two Dice

R W	R W	R W	R W	R W	R W
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

In the table we are using an ordered pair notation. For example, (3,4) means a 3 on the red die and a 4 on the white die. Notice that this is quite different from (4,3), a 4 on the red die and a 3 on the white.

Sometimes the possible outcomes of an experiment are represented in a sample space as shown below.



The circled × corresponds to the outcomes (3,4). To each of the 36 x's in the sample space we have assigned the probability of $\frac{1}{36}$ since each outcome is equally likely to all others. Thus $P(3,4) = \frac{1}{36}$.

With a sample space of this type, many probability problems reduce themselves to simple problems of counting applied to the formula $P(E) = \frac{t}{n}$. This relationship of counting to probability is very important and is one of the reasons why probability makes an appropriate topic for the junior high school mathematics class.

If we ask for the probability of getting a sum of 3 on one roll of the red and white dice, there are two possibilities associated with the event, (1,2) and (2,1). The probability then is given by

$$P(\text{sum of } 3) = P\left[(1,2) \text{ or } (2,1)\right] = \frac{2}{36} .$$

Notice, however, that each individual event has a probability of $\frac{1}{36}$

so that we could have arrived at the same answer by adding the individual probabilities.

$$P\Big((1,2) \text{ or } (2,1)\Big) = P(1,2) + P(2,1) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$$

This property of adding probabilities holds only when the outcomes under question are mutually exclusive: that is, when they cannot occur at the same time. If events A and B are mutually exclusive and have probabilities $P(A)$ and $P(B)$ respectively, then

$$P(A \text{ or } B) = P(A) + P(B) .$$

Consider again the probabilitiy of getting a sum of 3 or less on a single roll of the red and white dice. The sum of 3 or less means a sum of 3 or a sum of 2. (Note that 2 is the lowest sum possible on two dice.) The event 3 and the event 2 are mutually exclusive, hence we proceed as follows.

$$\begin{aligned} P(\text{sum of } 3 \text{ or less}) &= P(\text{sum of } 3 \text{ or sum of } 2) \\ &= P(\text{sum of } 3) + P(\text{sum of } 2) \\ &= P\Big((1,2) \text{ or } (2,1)\Big) + P(1,1) \\ &= P(1,2) + P(2,1) + P(1,1) \\ &= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} \\ &= \frac{3}{36} = \frac{1}{12} \end{aligned}$$

Our result, of course, agrees with that found for the same problem solved directly by counting points in the sample space.

Class Exercises

Use the table developed in this section to answer Exercises 15-18.

15. What is the probability of a result with a sum of 8?

16. What is the probability of getting a double? (both faces the same)

17. What is the probability of getting a double or a sum of nine?

18. What is the probability of getting a double or a sum of eight?

19. From a bag containing 3 red marbles, 5 white marbles, and 4 black marbles, one is drawn. Answer the following questions.
    (a) What is the probability of getting a red? white? black?
    (b) What is the probability of getting a red or black?
    (c) What is the probability of getting a white or red?
    (d) What is the probability of getting a red or white or black?

14.5 Probability of A and B

The question considered in the last section, the probability of either A or B, has its counterpart which may be asked as follows: "What is the probability of both events A and B occurring?" If we consider the simple case of flipping two coins, then we have four possible outcomes:

$$(H,H)\ ,\quad (H,T)\ ,\quad (T,H)\ ,\quad (T,T)\ .$$

Again, we adopt the notation where the first letter corresponds to the first coin; the second to the second coin. We agree that each of these four possible outcomes is equally likely to each other. Hence to each we assign the probability $\frac{1}{4}$. Thus, we may write

$$P(H,H) = \frac{1}{4}\ .$$

If A is the event that the first coin shows heads and B the event that the second coin shows heads, then we have $P(A \text{ and } B) = \frac{1}{4}$.
Notice, however, that individually

$$P(A) = \frac{1}{2} \quad \text{and} \quad P(B) = \frac{1}{2}\ .$$

In this case then,

$$P(A \text{ and } B) = P(A) \cdot P(B)\ .$$

In other words,

$$\begin{aligned} P(H,H) &= P\Big((\text{first coin } H) \text{ and } (\text{second coin } H)\Big) \\ &= P(\text{first coin } H) \cdot P(\text{second coin } H) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4}\ . \end{aligned}$$

Let us try this approach on the probability of getting a (1,1) when rolling the red and white dice. We already know that this probability is $\frac{1}{36}$ but notice again that the probability of each individual event (a 1 on the red die and a 1 on the white die) is $\frac{1}{6}$ so that the desired probability is given by the product $\frac{1}{6} \cdot \frac{1}{6}$.

This observation is true in general whenever the events are independent. By independent we mean that the outcome of one event has no effect on the outcome of the second. In general:

> If events A and B are independent, with probabilities $P(A)$ and $P(B)$ respectively, then the probability that both events occur is given by
>
> $$P(A \text{ and } B) = P(A) \cdot P(B)\ .$$

As an example of the above, suppose we flip a coin and roll a single die and ask the probability of getting both a head and a 5. Certainly the two events are independent, and since the probabilities of the two events are $\frac{1}{2}$ and $\frac{1}{6}$ respectively, we have

$$\begin{aligned} P(H \text{ and } 5) &= P(H) \cdot P(5) \\ &= \frac{1}{2} \cdot \frac{1}{6} \\ &= \frac{1}{12} . \end{aligned}$$

This example is simple and could also be solved by a tree or table showing all possible outcomes. In more complicated examples, however, the use of the individual probabilities is simpler.

Class Exercises

20. Find the probability of a head showing on each of 5 tosses of a coin.

21. A coin is tossed and a die is rolled. What is the probability of getting a head and an odd number?

22. In the preceding problem what is the probability of getting a head and a number less than six?

Answers to Class Exercises

1. $12\frac{1}{2}$ %

2. 135°

3. 45°

6. mean = 7
median = 6
mode = 5

7. range is 7

8. average deviation = $2\frac{2}{11}$

9. $\frac{1}{2}$

10. $\frac{1}{2}$

11. $\frac{1}{13}\ (\frac{4}{52})$

12. $\frac{1}{26}\ (\frac{2}{52})$

13. $\frac{1}{2}\ (\frac{4}{8})$

14. $\frac{1}{4}\ (\frac{2}{8})$

15. $\frac{5}{36}$

16. $\frac{1}{6}\ (\frac{6}{36})$

17. $\frac{1}{6} + \frac{4}{36} = \frac{5}{18}$

18. $\frac{5}{18}\ (\frac{10}{36})$ Note that these events are not mutually exclusive. (4,4) gives a sum of 8 and is at the same time a double.

19. (a) $\frac{3}{12}$, $\frac{5}{12}$, $\frac{4}{12}$ (b) $\frac{7}{12}$ (c) $\frac{2}{3}\ (\frac{8}{12})$ (d) 1

20. $\frac{1}{32}$

21. $\frac{1}{4}$

22. $\frac{5}{12}$

Chapter Exercises

1. Six darts were thrown at a circular dart board. The following observations were obtained: $(8, 7\frac{1}{2}, 3\frac{1}{2}, 4, 5, 4)$. Each observation is the measured distance in inches of a dart from the center of the target.

   Find the mean, median, and mode of the data.

2. What is the range and average deviation of the data in Exercise 1?

3. Suppose in Exercise 1, all the measures had been doubled by mistake. What would happen to the mean? Is your conclusion true in general?

4. What would happen to the range and average deviation if the data in Exercise 1 had been doubled? Are your conclusions true in general?

5. If a bag of 100 oranges contains 9 bad, what is the probability of the first orange chosen being good? If you have given away 37 oranges, one of which was bad, what is the probability that the next one is bad?

6. Make a table showing the possible outcomes in flipping four coins.

Using the table above, answer Exercises 7-10.

7. What is the probability of all heads?

8. What is the probability of exactly three heads?

9. What is the probability of one or more heads. (Hint: First find the probability of no heads.)

10. What is the probability of exactly one head or all tails?

11. An ordinary deck of 52 playing cards is shuffled, one card drawn, replaced, the deck shuffled, and a second card drawn.
    (a) What is the probability that both cards are red?
    (b) What is the probability that the first card is a spade and the second is the ace of hearts?
    (c) What is the probability that the second card is the same as the first?
    (d) What is the probability that the two of hearts is chosen first and the three of hearts is chosen second?

12. A coin is flipped ten times and a head appears each time. Assuming the coin to be honest, what is the probability of a head appearing next. (Hint: The coin does not have a memory.)

ANSWERS TO CHAPTER EXERCISES

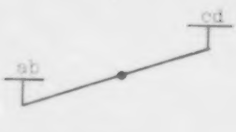
Answers to Problems in the Introduction

1.
$$\begin{array}{rcr} \text{SEND} & \text{———} & 9567 \\ +\ \text{MORE} & \text{———} & 1085 \\ \hline \text{MONEY} & \text{———} & 10652 \end{array}$$

2. The winning strategy for the sample number game is to always choose a number so that the total after your turn will always be a multiple of seven.

   The second number game, where the winning sum is 85, can be won by the player reaching 77 first. Regardless of what number his opponent then calls, he will always be able to choose a number that will make the sum 85. The magic number is eight, one more than the largest number that can be used. Since $85 = 8 \cdot 10 + 5$ all critical points are 5 more than multiples of 8 ; i.e., $77 = 72 + 5$, $69 = 64 + 5$, and so forth. The first player may win by choosing 5 and then each time afterward, picking a number to make the total 5 more than a multiple of eight; i.e, 13, 21, 29, ...

3. The problem of the counterfeit coin among six coins differs from the heavy marble problem in that weighing three coins against three coins will yield no new information. From the fact that the scale does not balance we only learn that the coins are not all of the same weight. To do this problem it will be convenient to label the coins a, b, c, d, e, f. Suppose we weight a, b against c, d. There are three possible outcomes of such a weighing.



Case I

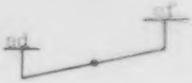
Case II

Case III

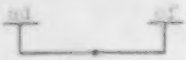
Case I

Here we see that either a or b is heavy or c or d is light, while e and f are good.

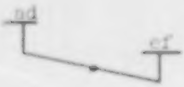
Second weighing -- compare a, d with e, f. Again three outcomes are possible.



thus a is heavy
(problem solved)

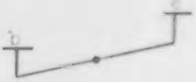


thus b is heavy
or c is light
(third weighing needed)

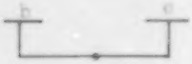


thus d is light
(problem solved)

Third weighing -- compare b and e -- two possibilities exist.



thus b is heavy
(problem solved)



thus c is light
(problem solved)

Case II

When a, b and c, d balance, either e or f is the false coin.

Second weighing -- compare e and a . Three possibilities exist.

thus e is light
(problem solved)

thus f is the
counterfeit coin
(third weighing
needed to determine
if heavy or light.)

thus e is heavy
(problem solved)

Third weighing -- compare f and a. Two possibilities exist.

thus f is light
(problem solved)

thus f is heavy
(problem solved)

Case III

This case is handled with the same reasoning as Case I with the obvious changes. Start with a or b in light and c or d is heavy. Notice that the key to the solutions of both Case I and Case III rests on the interchanging of the position of a coin from one side to the other.

4. Only figures b and c may be traced. Figures a and d each have more than two odd vertices.

5. It is not possible to connect the utilities and houses under the conditions stated. This is discussed further in Chapter 10.

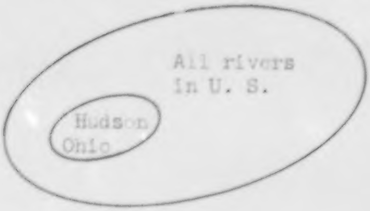
6. The smallest possible number of weights needed to weigh objects, in pounds, between 1 pound and 63 pounds is six; namely, weights of 1, 2, 4, 8, 16, and 32 pounds. A similar problem occurs again in Chapter 2.

Chapter 1

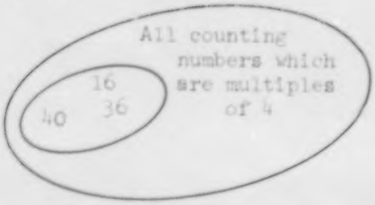
Answers to Chapter Exercises

1. $\{4\}$, $\{5\}$, $\{6\}$, $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, $\{4,5,6\}$, $\emptyset$.

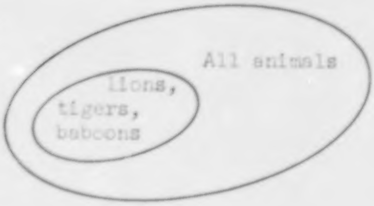
2. (a)


All rivers
in U. S.
Hudson
Ohio

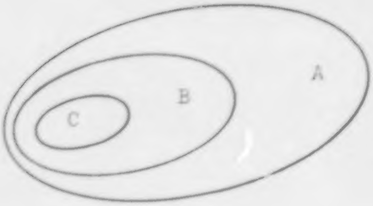
(c)


All counting
numbers which
are multiples
of 4
16
36
40

(b)


All animals
lions,
tigers,
baboons

3. Yes.



4. (a) $A \cap B = \{\text{girl}, \text{chair}\}$

(b) $A \cap C = \{\text{chair}\}$ and $C \cap A = \{\text{chair}\}$

(c)
$$\begin{aligned} A \cap (B \cup C) &= \{\text{boy}, \text{girl}, \text{chair}\} \cap (\{\text{girl}, \text{chair}, \text{dog}\} \cup \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}, \text{cat}\} \\ &= \{\text{girl}, \text{chair}\} \end{aligned}$$

also
$$\begin{aligned} (A \cap B) \cup (A \cap C) &= (\{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}\}) \cup (\{\text{boy}, \\ &\qquad \text{girl}, \text{chair}\} \cap \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{girl}, \text{chair}\} \cup \{\text{chair}\} \\ &= \{\text{girl}, \text{chair}\} \end{aligned}$$

(d)
$$\begin{aligned} A \cap (B \cap C) &= \{\text{boy}, \text{girl}, \text{chair}\} \cap (\{\text{girl}, \text{chair}, \text{dog}\} \cap \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{chair}, \text{dog}\} \\ &= \{\text{chair}\} \end{aligned}$$

also
$$\begin{aligned} C \cap (A \cap B) &= \{\text{chair}, \text{dog}, \text{cat}\} \cap (\{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}\}) \\ &= \{\text{chair}, \text{dog}, \text{cat}\} \cap \{\text{girl}, \text{chair}\} \\ &= \{\text{chair}\} \end{aligned}$$

5. (a) Yes.

$$\emptyset \cup H = H$$
$$H \cup \emptyset = H$$

Therefore, $\emptyset \cup H = H \cup \emptyset$.

(b) No. $\emptyset \cap H = \emptyset$. $\emptyset$ has no elements in common with H.

6. (a) Yes. The counting numbers are composed of the even numbers and the odd numbers.

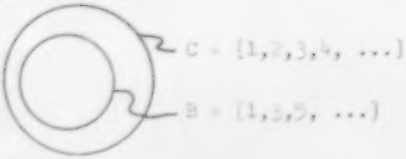
(b) Yes. Same reason as (a).

(c) No. A and B are disjoint subsets of C.

(d) Yes. The set of elements in A or B is the same as those in B or A.

(e) No. Same reason as (c).

(f)



(g) No. They do not contain the same elements.

7. (a) $F = \{1,3,5,7,9, \ldots\}$. F is the set of odd counting numbers.

(b) $\emptyset$

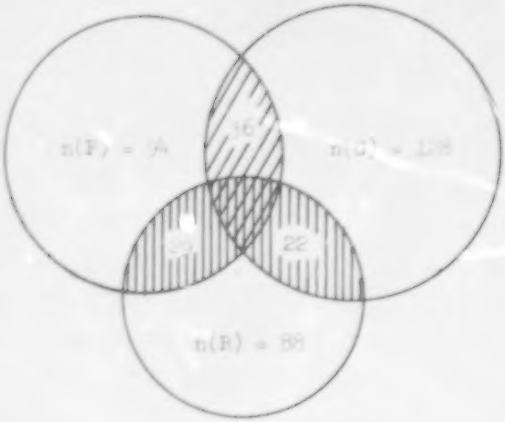
(c) Yes. Set F could be, for example, the same as set C.

8. (a) Yes. Every element of A is an element of B.

(b) Yes. Same reason as (a).

9. No. You can look for empty spaces.

10.



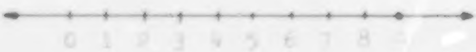
The diagram simplifies our bookkeeping and also shows that the intersection of the sets is included three times but should not be subtracted three times.

11. The columns on a calendar establish such a correspondence.

12. (a) $\{3,4,5,6,7\}$

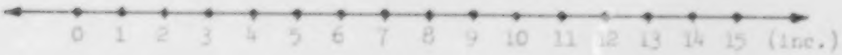
0
1
2
3
4
5
6
7
8
9

(b) $\{9\}$

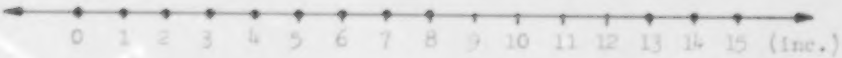


12. (continued)

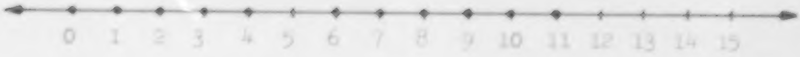
(c) {0,1,2,3, ...} , the set of whole numbers.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 (inc.)

(d) The set of all whole numbers except 9, 10, 11, 12.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 (inc.)

(e) {0,1,2,3,4,6,7,8,9,10,11}, the set of all whole numbers less than 12 except 5.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

13.
$$\begin{array}{llllllllllll} W = \{ & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots\} & \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & & & & \text{etc.} \\ O = \{ & 1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & \ldots\} & \end{array}$$

Set O is a proper subset of set W, yet for any member of W we may find a corresponding member of O, and for any member of O we may find a corresponding member of W. In other words, given an infinite set, a proper subset (which is itself infinite) may be put into a one-to-one correspondence with the given set.

Chapter 2

Answers to Chapter Exercises

1. (a) 𓎆𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺𓏺

34_{five}

25_{seven}

(b) 𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺

203_{five}

104_{seven}

(c) 𓍢𓍢𓍢𓍢𓍢𓍢 𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺

10131_{five}

1641_{seven}

(d) 𓆼 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢 𓎆𓎆𓎆𓎆𓎆𓎆

30320_{five}

5500_{seven}

2. (a) $(1 \times 5^2) + (0 \times 5) + (0 \times 1)$

(b) $(1 \times 2^3) + (1 \times 2^2) + (1 \times 2) + (0 \times 1)$

(c) $(2 \times 3^2) + (0 \times 3) + (1 \times 1)$

(d) $(1 \times 4^2) + (1 \times 4) + (1 \times 1)$

(e) $(1 \times 8) + (7 \times 1)$

(f) $(T \times 1)$ or (10×1)

(g) $(1 \times 10^2) + (0 \times 10) + (0 \times 1)$

(h) $(1 \times 7^2) + (0 \times 7) + (0 \times 1)$

3. (a) 44_{five} (e) 10_{eight}

(b) 1101_{two} (f) 9_{twelve}

(c) 200_{three} (g) 99

(d) 103_{four} (h) 66_{seven}

4. (a) 1 2 2 122_{five} (d) 0 3 0 30_{five}

(b) 0 4 4 44_{five} (e) 2 2 3 223_{five}

(c) 3 4 4 344_{five} (f) 4 4 4 444_{five}

5.

Base ten Numerals	Base five Numerals	Base ten Numerals	Base five Numerals
1	/	14	∠□
2	∠	15	△○
3	△	16	△/
4	□	17	△∠
5	/○	18	△△
6	//	19	△□
7	/∠	20	□○
8	/△	21	□/
9	/□	22	□∠
10	∠○	23	□△
11	∠/	24	□□
12	∠∠	25	/○○
13	∠△		

6. (a) 294 (b) 193 (c) 1511 (d) 525

7. (a) 3145_{six} (b) 101100_{two} (c) $10E_{\text{twelve}}$ (d) 11031_{four}

8. 73_{eight}

Chapter 3

Answers to Chapter Exercises

1. (a) 432_{eight} (d) 710_{eight}
   (b) 1337_{eight} (e) 303_{eight}
   (c) 404_{eight}

2. (a) 40_{eight} (b) 611_{eight} (c) 12_{eight}

3. (a) 3237_{eight} (b) 34136_{eight}

4. 7_{eight}

   362_{eight}

5. (a) $9 = 72 \div 8$ or $8 = 72 \div 9$
   $2 = 80 \div 40$ $40 = 80 \div 2$
   $5 = 25 \div n$ $n = 25 \div 5$
   $4 = 24 \div n$ $n = 24 \div n$
   $n = 100 \div 10$ $10 = 100 \div n$

   (b) $3_{\text{five}} = 11_{\text{five}} \div 2_{\text{five}}$ or $2_{\text{five}} = 11_{\text{five}} \div 3_{\text{five}}$
   $n_{\text{five}} = 31_{\text{five}} \div 4_{\text{five}}$ or $4_{\text{five}} = 31_{\text{five}} \div n_{\text{five}}$

   (c) $4_{\text{seven}} = 26_{\text{seven}} \div 5_{\text{seven}}$ or $5_{\text{seven}} = 26_{\text{seven}} \div 4_{\text{seven}}$
   $4_{\text{seven}} = 33_{\text{seven}} \div n_{\text{seven}}$ or $n_{\text{seven}} = 33_{\text{seven}} \div 4_{\text{seven}}$
   $n_{\text{seven}} = 100_{\text{seven}} \div n_{\text{seven}}$

6. 6_{seven}, 10_{seven}

7. (a) base five (d) base five
   (b) base seven (e) any base > 3
   (c) base seven (f) any base > 8

8. (a) 0

(b) 0, 1, or 2. Evenness cannot be recognized by the last digit in base three.

9. (a) 0 or 1 (b) 0 or 1

10. It is not possible under the conditions stated, since N + T = N implies T = zero and E + H = E implies H = zero. Both cannot be zero, under the conditions stated.

11. (a) 140_{seven} (b) 3_{seven} (c) 462_{seven}

Chapter 4

Answers to Chapter Exercises

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. (a) 1 (b) 2 (c) 2 (d) 1 (e) 3

3. (a) yes

(b) yes

(c) yes

(d) The inverse of 1 is 1, of 3 is 3.

(e) Not necessarily. $2 \times 2 = 0$.

4. (a) Both are commutative.

(b) Yes, 0

(c) yes

(d) No. $0 + (E \times 0) = 0 + E = 0$, but $(0 + E) \times (0 + 0) = 0 \times E = E$

5. Not closed; $2 \cdot 25 + 3 = 53$ and 53 is not in the set.
Not commutative.
Not associative.

6. Closed.
Not commutative.
Not associative.

7. (a)

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

(b) Yes. (c) Yes. (d) Yes. (e) Yes, I. (f) Yes. Each element is its own inverse.

8. (a) Yes.

$$A \cup (B \cap C) = \{1,2,3,4,5\} \cup \{3,5,8\} = \{1,2,3,4,5,8\}$$

$$(A \cup B) \cap (A \cup C) = \{1,2,3,4,5,7,8\} \cap \{1,2,3,4,5,8,9\} = \{1,2,3,4,5,8\}$$

(b) Yes.

$$A \cap (B \cup C) = \{1,2,3,4,5\} \cap \{1,3,4,5,7,8,9\} = \{1,3,4,5\}$$

$$(A \cap B) \cup (A \cap C) = \{3,4,5\} \cup \{1,3,5\} = \{1,3,4,5\}$$

Chapter 5

Answers to Chapter Exercises

1. This may be shown directly by simply computing both sides. A more interesting and illuminating method is through the use of the distributive law.

$$\begin{aligned}4(5 + 6 + 7 + 8) &= 4((5 + 6 + 7) + 8)\\ &= 4(5 + 6 + 7) + (4 \cdot 8)\\ &= 4((5 + 6) + 7) + (4 \cdot 8)\\ &= 4(5 + 6) + (4 \cdot 7) + (4 \cdot 8)\\ &= (4 \cdot 5) + (4 \cdot 6) + (4 \cdot 7) + (4 \cdot 8)\end{aligned}$$

2. This may also be shown directly by computation but it is instructive to see a method using the associative law.

$$\begin{aligned}\Big(\big((3 + 4) + 9\big) + 5\Big) + 2 &= \big((3 + 4) + 9\big) + (5 + 2)\\ &= (3 + 4) + \big(9 + (5 + 2)\big)\\ &= 3 + \Big(4 + \big(9 + (5 + 2)\big)\Big)\end{aligned}$$

3. There are many answers: $\frac{102}{128}$ and $\frac{459}{576}$ are two possible answers.

4. There is nothing imporper about improper fractions. An improper fraction, numerator greater than the denominator, may be written in another form, e.g., $\frac{19}{3}$ as $6\frac{1}{3}$. This second form may serve a useful psychological purpose and may be better suited to some calculations or commerce, but serves no useful mathematical purpose. Indeed there are situations in which such proper forms are a hindrance.

5. Yes. The symbol $\frac{11}{5}$ names a solution to $5x = 11$. Since it can be shown that the equations $5x = 11$ and $35x = 77$ are the same, $\frac{11}{5}$ must also name the solution of $35x = 77$.

   No. The symbol $\frac{7}{9}$ names a solution to $9x = 7$. Since $9x = 7$ and $21x = 27$ are not the same equation, $\frac{7}{9}$ does not name a solution of $21x = 27$.

6. In each case apply the definition for equivalent fractions.

7. 6, 9, 103.

8. Yes, in each case. The counting numbers are 3, 2, 31, and a. Observe, for example, that while $\frac{6}{2}$ is an answer of $2x = 6$, so also is 3 a solution of this equation.

9. $\frac{3}{17}$, $\frac{3}{8}$, $\frac{3}{6}$, $\frac{3}{5}$, $\frac{3}{2}$, $\frac{3}{1}$.

10. $\frac{2}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, $\frac{7}{7}$, $\frac{9}{7}$, $\frac{14}{7}$, $\frac{18}{7}$.

11. $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, $\frac{19}{20}$, $\frac{99}{100}$

12. b, d, e.

Chapter 6

Answers to Chapter Exercises

1. (a) $\frac{193}{120}$ (b) $\frac{17}{120}$ (c) $\frac{77}{120}$ (d) $\frac{105}{88}$

2. (a) $\frac{3}{2}$ (b) $\frac{7}{4}$ (c) $\frac{15}{8}$ (d) $\frac{31}{16}$

3. (a) $\frac{127}{64}$ (b) $\frac{2047}{1024}$

4. $\frac{60}{36}$, $\frac{30}{72}$. The answers are not the same which shows that division is not associative.

5. $\frac{18}{4}$, $\frac{{}^{-}3}{8}$. The answers are not the same which shows that division is not distributive over subtraction.

6. The problem is to show that the two expressions, $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f}$ and $\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$ are equal.

$$(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} = (\frac{ac}{bd}) \cdot \frac{e}{f}$$

$$= \frac{ace}{bdf} .$$

Also,

$$\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f}) = \frac{a}{b} \cdot (\frac{ce}{df})$$

$$= \frac{ace}{bdf} .$$

Thus, since the two expressions are each equal to $\frac{ace}{bdf}$ we see that the associative law does hold for multiplication of rational numbers.

7. (a) ${}^{-}4$ (b) 22 (c) ${}^{-}22$ (d) 4

8. (a) $\frac{{}^{-}30}{8}$ (b) $\frac{20}{28}$ (c) $\frac{3}{8}$ (d) ${}^{-}\frac{12}{15}$

9. (a) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ (c) $\frac{1}{4} + \frac{1}{5}$

   (b) $\frac{1}{3} + \frac{1}{5}$ (d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{30}$

10. (a) $\frac{1}{1 \cdot 2} = \frac{1}{2}$ and $\frac{1}{1} - \frac{1}{2} = \frac{2 - 1}{2} = \frac{1}{2}$

(b) $\frac{1}{2 \cdot 3} = \frac{1}{6}$ and $\frac{1}{2} - \frac{1}{3} = \frac{3 - 2}{6} = \frac{1}{6}$

(c) $\frac{1}{4 \cdot 5} = \frac{1}{20}$ and $\frac{1}{4} - \frac{1}{5} = \frac{5 - 4}{20} = \frac{1}{20}$

(d) $\frac{1}{9 \cdot 10} = \frac{1}{9} - \frac{1}{10}$

(e) $\frac{1}{18 \cdot 19} = \frac{1}{18} - \frac{1}{19}$

(f) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{18 \cdot 19} = (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{18} - \frac{1}{19})$

$= \frac{1}{1}({}^{-}\frac{1}{2} + \frac{1}{2}) + ({}^{-}\frac{1}{3} + \frac{1}{3}) + \dots + ({}^{-}\frac{1}{18} + \frac{1}{18}) - \frac{1}{19}$

$= \frac{1}{1} - \frac{1}{19}$

$= \frac{18}{19}$

11. (a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{18}{19}$ (e) $\frac{99}{100}$

12. $\frac{175}{144}$

13.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{7}{12}$
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$

Chapter 7

Answers to Chapter Exercises

1. (a) $39 = 3 \times 13$ (e) $180 = 2^2 \times 3^2 \times 5$
   (b) $60 = 2^2 \times 3 \times 5$ (f) $258 = 2 \times 3 \times 43$
   (c) $81 = 3^4$ (g) $576 = 2^6 \times 3^2$
   (d) $98 = 2 \times 7^2$ (h) $2324 = 2^2 \times 7 \times 83$

2. (a) l.c.m. = 78 (b) l.c.m. = 210 (c) l.c.m. = 1517
   g.c.f. = 6 g.c.f. = 7 g.c.f. = 1

3.

N	Factors of N	Number of Factors	Sum of Factors
9	1, 3, 9	3	13
10	1, 2, 5, 10	4	18
11	1, 11	2	12
12	1, 2, 3, 4, 6, 12	6	28
13	1, 13	2	14
14	1, 2, 7, 14	4	24
15	1, 3, 5, 15	4	24
16	1, 2, 4, 8, 16	5	31
17	1, 17	2	18
18	1, 2, 3, 6, 9, 18	6	39
19	1, 19	2	20
20	1, 2, 4, 5, 10, 20	6	42
21	1, 3, 7, 21	4	32
22	1, 2, 11, 22	4	36
23	1, 23	2	24
24	1,2,3,4,6,8,12,24	8	60
25	1, 5, 25	3	31
26	1, 2, 13, 26	4	42
27	1, 3, 9, 27	4	40
28	1, 2, 4, 7, 14, 28	6	56
29	1, 29	2	30
30	1,2,3,5,6,10,15,30	8	72

(a) 2,3,5,7,11,13,17,19,23,29 (the prime numbers)

(b) 4,9,25 (the squares of prime numbers)

(c) Three: 1, p and p^2 (d) Four: 1,p,q,pq. The sum is 1+p+q+pq.

(e) The factors are: $1, 2, 2^2, 2^3, \ldots, 2^k$. There are k + 1 of them.

4. (a) No. It is not possible to have exactly four numbers between two odd numbers. Between any two odd primes there is always an odd number of numbers. If they are consecutive odd primes all the numbers between would have to be composite.

(b) Yes. For example, between 23 and 29 there are exactly 5 composite numbers: 24, 25, 26, 27, 28.

5. (a) 135, 222, 783, and 1065 are all divisible by three.

(b) 222 is the only number divisible by six.

(c) 135 and 783 are divisible by nine.

(d) 135 and 1065 are divisible by five.

(e) 135 and 1065 are divisible by fifteen.

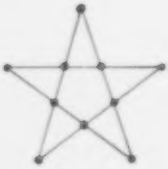
(f) None of the numbers are divisible by four.

6.

Rows	Bulbs per row
1	112
2	56
4	28
8	14
16	7

(Bulbs and rows may be interchanged.)

7. The pattern is a five-pointed star.



8. (a) No

(b) Yes

(c) No

(d) Yes

Chapter 8

Answers to Chapter Exercises

1. (a) $(3 \times 10) + (2 \times 1) + (7 \times \frac{1}{10}) + (8 \times \frac{1}{10^2}) + (5 \times \frac{1}{10^3})$

   (b) $(4 \times 5) + (2 \times 1) + (3 \times \frac{1}{5}) + (4 \times \frac{1}{5^2}) + (1 \times \frac{1}{5^3})$

2. a, c

3. (a) $0.27 + 0.47 = (27 \times \frac{1}{100}) + (47 \times \frac{1}{100})$

   $= (27 + 47) \times \frac{1}{100}$

   $= 74 \times \frac{1}{100}$

   $= \frac{74}{100}$

   $= .74$

   (b) $0.4 \times 0.37 = (4 \times \frac{1}{10}) \times (37 \times \frac{1}{100})$

   $= (4 \times 37) \times (\frac{1}{10} \times \frac{1}{100})$

   $= 148 \times \frac{1}{1000}$

   $= \frac{148}{1000}$

   $= .148$

4. (a) $\frac{21}{4}$ (b) $\frac{46}{5}$

5. AC = 15, DE = 4, EF = 8

6. (a) 1000 % (b) 100 % (c) 10 % (d) 1 % (e) $\frac{1}{10}$ %

7. (a) 75 % (b) $133\frac{1}{3}$ % (c) $42\frac{6}{7}$ % (d) $6\frac{2}{3}$ % (e) $1\frac{2}{3}$ %

Chapter 9

Answers to Chapter Exercises

1. (a) $.66\overline{6}\ldots$ (b) $.44\overline{4}\ldots$ (c) $.27\overline{27}\ldots$ (d) $.02\overline{02}\ldots$

2. (a) 35353 (b) 35553 (c) 35535

3. (a) $\frac{4}{33}$ (b) $\frac{16}{37}$ (c) $\frac{7}{10}$

4. rational: b, e
   irrational: a, c, d

5. rational: a, c, e
   irrational: b, d

6. c; a

7. Answers will vary.
   (a) rational: 0.345335 ; $0.34534\overline{34}\ldots$
   (b) irrational: 0.34534533345333 ... ; 0.3453373337 ...

8. $\frac{1}{7} = .142857\overline{142857}\ldots$ $\frac{4}{7} = .571428\overline{571428}\ldots$
   $\frac{2}{7} = .285714\overline{285714}\ldots$ $\frac{5}{7} = .714285\overline{714285}\ldots$
   $\frac{3}{7} = .428571\overline{428571}\ldots$ $\frac{6}{7} = .857142\overline{857142}\ldots$

   Note that the same digits appear in each representation, (1,4,2,8,5,7). These then reappear in cyclic fashion for each decimal, with the initial digits being in the order 1, 2, 4, 5, 7, 8.

9. $\frac{1}{13} = .\overline{076923}$ $\frac{2}{13} = .\overline{153846}$
   $\frac{3}{13} = .\overline{230769}$ $\frac{4}{13} = .\overline{307692}$
   $\frac{5}{13} = .\overline{384615}$ $\frac{6}{13} = .\overline{461538}$
   $\frac{7}{13} = .\overline{538461}$ $\frac{8}{13} = .\overline{615384}$
   $\frac{9}{13} = .\overline{692307}$ $\frac{10}{13} = .\overline{769230}$
   $\frac{11}{13} = .\overline{846153}$ $\frac{12}{13} = .\overline{923076}$

10. (a) 1,2 (e) 2,3
    (b) 11,12 (f) 9,10
    (c) 3,4 (g) 14,15
    (d) 5,6 (h) 8,9

11. (a) $30\frac{1}{3}$

(b) $\sqrt{3}$

(c) 24

(d) $\frac{1}{28}$

(e) 5

(f) 5

(g) $7\frac{7}{8}$

(h) $\sqrt{11}$

(i) 18

(j) 2

(k) 33

(l) $\sqrt{29}$

(m) 16

(n) $15\frac{10}{11}$

(o) $\sqrt{13}$

(p) $\sqrt{5}$

counting numbers : c, e, f, i, j, k, m

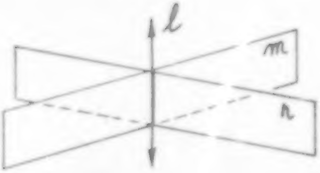
rational numbers : a, d, g, n

irrational numbers : b, h, l, o, p

Chapter 10

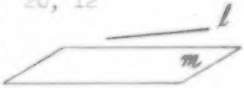
Answers to Chapter Exercises

1.

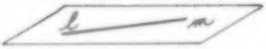


2. (a) 8, 0
   (b) 20, 12

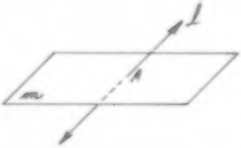
3. (a)



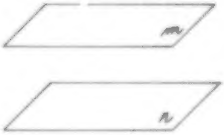
(b)



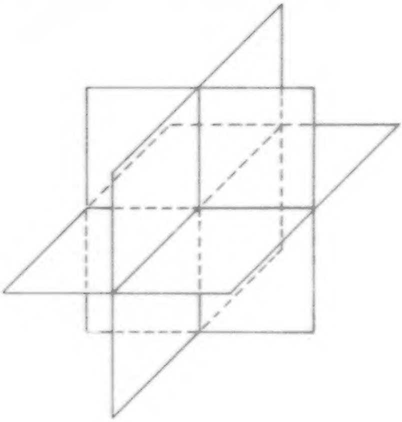
(c)



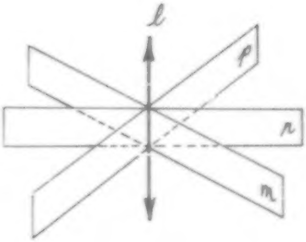
4. (a) See Exercise 1.
   (b)



(d)



(c)



5. The ray includes an end point, the half-line does not.

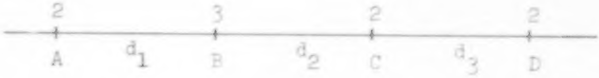
6. AB denotes the line passing through points A and B.
$\overline{AB}$ is the segment with A and B as end points.
BA is the ray starting at point B and passing through A.
AB is the ray starting at point A and passing through B.

7. Many answers are possible, only one set is given.

(a) ABF, FBC
(b) HEF, DAB
(c) HEF, ABF, FBC
(d) ABF, HDB, FBC
(e) EA, FB
(f) HF, AC
(g) AB, BD, BC
(h) HF, EG, LM
(i) EAB, HDB, FBC, DAB

8. One plane, if the point is not on the line.

9. At first not enough information seems to be given. How far apart are the houses? The total distance walked, and thus the minimum distance, would seem to depend upon the distances between each house. Let us start, however, and for the moment assign distances between houses as shown.



Then if meetings are held at house A, 7 boys must walk distance d_1, 4 must walk d_2, and 2 must walk d_3, so that the total distance walked is

$$7d_1 + 4d_2 + 2d_3 \qquad \text{(house A)}$$

Using the same argument gives the following:

$$2d_1 + 4d_2 + 2d_3 \qquad \text{(house B)}$$

$$2d_1 + 5d_2 + 2d_3 \qquad \text{(house C)}$$

$$2d_1 + 5d_2 + 7d_3 \qquad \text{(house D)}$$

Examining the four cases shows that meeting at house B will minimize walking. Surprisingly enough, the conclusion is the same regardless of the distances d_1, d_2, and d_3.

Chapter 11

Answers to Chapter Exercises

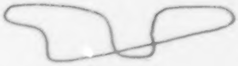
1. The models form a rectangular prism and a hexagonal prism.

2. a and d are closed, only a is a simple closed curve.

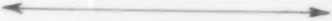
3. The angles are not equal, since they are two different angles. Recall that angles are sets of points and that sets are equal only when they are identical.

4. Answers may vary widely; only samples are given.

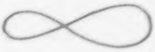
(a)



(b)



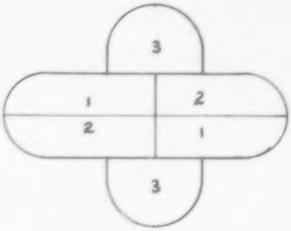
(c)

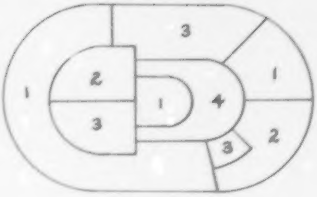


5. No. Euler's formula does not hold. $V = 12$, $E = 20$, $F = 9$, and $V - E + F = 1$.

6. If a Moebius strip with two twists is cut down the middle, it falls into two loops which are interlocked.

7.

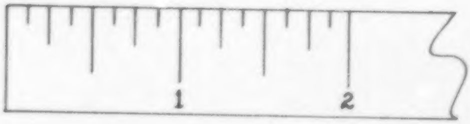




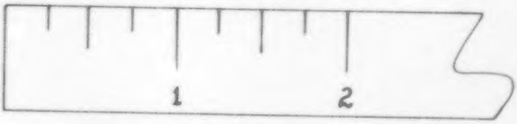
Chapter 12

Answers to Chapter Exercises

1. Subdivided to $\frac{1}{8}$ inch



2. Subdivided to $\frac{1}{4}$ inch



3. The dimensions of the larger rectangle should be $5\frac{1}{4}''$ by $3\frac{3}{4}''$.
   The dimensions of the smaller rectangle should be $4\frac{3}{4}''$ by $3\frac{1}{4}''$.
4. Square, rectangle, parallelogram, rhombus, trapezoid, "kite". (There may be others suggested.)
5. A polygon whose sides are congruent and whose angles are congruent.
6. Two circles are congruent if their radii are congruent.
7. They are perpendicular.
8. One radius is twice the other.
9. Given a point C and a distance r, a sphere is the set of all points in space at distance r from point C.
10. Not necessarily. The definitions will vary depending on what is considered a "line" or "segment" on a sphere, and intuitive definitions should be accepted. This is difficult to define because a "triangle" may have more than one right angle. Note: The purpose of this exercise is to cause you to consider the "ground rules" of plane geometry, and that these rules do not necessarily hold in another physical situation.

Chapter 13

Answers to Chapter Exercises

1. (a) 1 foot each (b) 45", 4'
   (c) 4 is not the sum of 1, 1, and 1. Even though each error was less than one-half foot, the sum of the errors was over half a foot and therefore must be counted in the measure of the perimeter.

2. The square has the greater area.

3. (a) 116 sq. units (b) 112.5 sq. units

4. (a) 90 cubic inches approximately.
   (b) Approximately 111.4 cubic inches.
   (c) Even though each error was less than one-half inch, the product of the three numbers in (b) would increase the volume measure significantly.

5. 4 feet

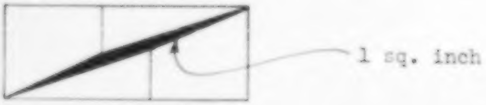
6. Approximately 0.004 cubic inches

7. (a) Approximately 13 inches
   (b) Approximately 38 cubic inches
   (c) Approximately 63 square inches

8. (a) $9\frac{1}{2}$", $7\frac{1}{2}$", $5\frac{1}{2}$" (c) $391\frac{7}{8}$ cu. in.
   (b) $10\frac{1}{2}$", $8\frac{1}{2}$", $6\frac{1}{2}$" (d) $580\frac{1}{8}$ cu. in.

9. (a) 64 sq. in. (b) 64 sq. in.

   This error is difficult to spot. Most people do not cut these exactly and miss the fact that there is a small parallelogram of area 1 square inch in the center of the completed rectangle. The figure might look similar to this:



10. (a) Circumference is doubled.
    (b) Area is 4 times as great.

Chapter 14

Answers to Chapter Exercises

1. mean = $5\frac{1}{3}$

   median = $4\frac{1}{2}$

   mode = 4

2. range = $4\frac{1}{2}$

   average deviation = $\frac{29}{18}$

3. The mean is also doubled.

$$m_1 = \frac{x_1 + x_2 + x_3 + \dots + x_n}{n}$$

$$m_2 = \frac{2x_1 + 2x_2 + 2x_3 + \dots + 2x_n}{n}$$

$$= 2\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) = 2m_1$$

The median is likewise doubled, for the middle element is still the middle element.

4. The range is also doubled for if x_a is the smallest and x_b is the largest in the original distribution the range is $x_b - x_a$. The new range will be $2x_b - 2x_a = 2(x_b - x_a)$.

The average deviation is doubled as the following example consisting of four elements will indicate.

$\{a, b, c, d\}$

$$m_1 = \frac{a + b + c + d}{4}$$

$$\text{Average deviation} = \frac{a - m_1 + b - m_1 + c - m_1 + d - m_1}{4}$$

$$= \frac{a + b + c + d - 4m_1}{4}$$

$\{2a, 2b, 2c, 2d\}$

$$m_2 = \frac{2(a + b + c + d)}{4} = 2m_1$$

$$\text{Average deviation} = \frac{2a + 2b + 2c + 2d - 4m_2}{4}$$

$$= \frac{2a + 2b + 2c + 2d - 8m_1}{4} = 2\left(\frac{a + b + c + d - 4m_1}{4}\right).$$

5. $\frac{91}{100}$, $\frac{8}{63}$

6. H H H H
H H H T
H H T H
H H T T
H T H H
H T H T
H T T H
H T T T
T H H H
T H H T
T H T H
T H T T
T T H H
T T H T
T T T H
T T T T

7. $P(H\ H\ H\ H) = \frac{1}{16}$

8. $P(\text{three } H\text{'s}) = \frac{1}{4}$

9. $$\begin{aligned} P(\text{at least one } H) &= 1 - P(\text{no } H) \\ &= 1 - P(T\ T\ T\ T) \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \end{aligned}$$

10. $P(\text{one } H \text{ or } T\ T\ T\ T)$
$= P(\text{one } H) + P(T\ T\ T\ T)$
$= \frac{1}{4} + \frac{1}{16}$
$= \frac{5}{16}$

11. (a) $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
(b) $\frac{1}{4} \times \frac{1}{52} = \frac{1}{208}$
(c) $\frac{1}{52}$
(d) $\frac{1}{52} \times \frac{1}{52} = \frac{1}{2704}$

12. $\frac{1}{2}$

GLOSSARY

Mathematical terms and expressions are frequently used with different meanings and connotations in different fields or levels of mathematics. The following glossary explains some of the mathematical words and phrases as they are used in this book. These are not intended to be formal definitions. More explanations as well as figures and examples may be found in the book by reference through the index.

A

ALGORITHM (ALGORISM). A special process for solving problems.

ANGLE. The union of two rays which have the same endpoint but which do not lie in the same line.

ARC. A part of a circle determined by two points on the circle.

AREA. A measurement in terms of a specified unit which is assigned to a closed region. Note that both number and unit must be given, as 30 square feet.

ASSOCIATIVE PROPERTY OF ADDITION. For the three numbers $\underline{a}$, $\underline{b}$, and $\underline{c}$

$$(a + b) + c = a + (b + c) .$$

ASSOCIATIVE PROPERTY OF MULTIPLICATION. For the three numbers $\underline{a}$, $\underline{b}$, and $\underline{c}$

$$(a \times b) \times c = a \times (b \times c) .$$

B

BASE (of a numeration system). The number used in the fundamental grouping. Thus 10 is the base of the decimal system and 2 is the base of the binary system.

BASE (of a geometric figure). A particular side or face of a geometric figure.

BINARY NUMERATION SYSTEM. A numeration system whose base is two.

BINARY OPERATION. An operation applied to a pair of numbers.

BRACES { }. Symbols used in this book exclusively to indicate sets of objects. The members of the set are listed or specified within the braces.

BROKEN LINE CURVE. A curve formed from segments joined end to end but not forming a straight line.

C

CIRCLE. The set of all points in a plane which are the same distance from a given point. A simple closed curve in a plane each of whose points is the same distance from a fixed point.

CLOSED CURVE. A curve that can be represented by a figure that starts and stops at the same point.

CLOSURE. An operation in a set has the property of closure if the result of the operation on members of the set is a member of the set.

COMMUTATIVE PROPERTY OF ADDITION. For the two numbers $\underline{a}$ and $\underline{b}$,

$$a + b = b + a .$$

COMMUTATIVE PROPERTY OF MULTIPLICATION. For the two numbers $\underline{a}$ and $\underline{b}$,

$$a \times b = b \times a .$$

COMPOSITE NUMBER. A whole number greater than 1 which is not a prime number.

CONE. A surface formed when a plane cuts a conical surface such that the intersection is a simple closed curve. The cone is that part of the conical surface between the vertex and the plane, the vertex, and the closed region cut from the plane that forms the base.

CONGRUENCE. The relationship between two geometric figures which have exactly the same size and shape.

CONVEX POLYGON. A polygon whose interior is in the interior of each of its angles. It is also defined as a polygon which lies entirely in or on the edge of the half plane determined by each of the sides in turn.

COUNTING NUMBERS. The numbers used in counting: $\{1, 2, 3, 4, 5, \ldots\}$

CURVE. A set of all those points which lie on a particular path from A to B.

CYLINDER. A surface formed when two parallel planes intersect a cylindrical surface. It is the portion of the cylindrical surface between the planes, together with the closed regions cut from the planes.

CYLINDRICAL SURFACE. A surface formed by all lines passing through a simple closed curve in a plane, parallel to a line not in the plane.

D

DECIMAL. A numeral written in the extended decimal place value system.

DECIMAL PLACE VALUE SYSTEM. A place value numeration system with ten as the base for grouping.

DEGREE. A common unit for numerical measure of angles. The symbol for a degree is $^\circ$.

DENSE. A property of the sets of rational and real numbers. The rational (real) numbers are dense because between any two rational (real) numbers there is a third rational (real) number.

DIAMETER OF A CIRCLE. A line segment which contains the center of the circle and whose endpoints lie on the circle.

DISJOINT SETS. Two or more sets which have no members in common.

DISTRIBUTIVE PROPERTY. A joint property of multiplication and addition. This property says that multiplication is distributive over addition. For any numbers $\underline{a}$, $\underline{b}$, and $\underline{c}$,

$$a \times (b + c) = (a \times b) + (a \times c) .$$

E

ELEMENT OF A SET. An object in a set; a member of a set.

EMPTY SET. The set which has no members.

EQUAL, symbol =. $A = B$ means that A and B are two different names for the same object.

EQUIVALENT NUMERALS. Numerals that name the same number.

EQUIVALENT SETS. Sets that can be put into a one-to-one correspondence.

EXPANDED FORM. 532 written as $(5 \times 10^2) + (3 \times 10) + (2 \times 1)$ is said to be written in expanded form.

F

FACTOR. If $bx = a$, with $\underline{a}$, $\underline{b}$, and $\underline{x}$, whole numbers, then $\underline{b}$ is a factor of $\underline{a}$.

FRACTION. Any expression of the form $\frac{x}{y}$ where x and y represent numbers.

G

GREATEST COMMON FACTOR. The largest whole number which is a factor of two or more given whole numbers.

H

HALF-LINE. A line separated by a point results in two half-lines, neither of which contains the point.

HALF-PLANE. A plane separated by a line results in two half-planes, neither of which contains the line.

HALF-SPACE. Space separated by a plane results in two half spaces, neither of which contains the plane.

I

IDENTITY ELEMENT FOR ADDITION. The number 0 which has the property

$$0 + a = a + 0 = a .$$

IDENTITY ELEMENT FOR MULTIPLICATION. The number 1 which has the property that

$$1 \times a = a \times 1 = a .$$

INTEGER. Any whole number or its opposite.

INTERSECTION OF TWO SETS. The set of all elements common to each of the given sets.

IRRATIONAL NUMBER. A real number which cannot be expressed in the form $\frac{a}{b}$ where $\underline{a}$ is an integer and $\underline{b}$ is a counting number, i.e., any number that is not a rational number.

L

LEAST COMMON MULTIPLE. The smallest non-zero whole number which is a multiple of each of two given whole numbers.

LENGTH OF A LINE SEGMENT. A measurement in terms of a specified unit which is assigned to the segment. Note that both number and unit must be given, as 3 feet or 5 miles, etc.

LINE (STRAIGHT LINE). A particular set of points in space (an undefined term in geometry). Informally it can be thought of as the extension of a line segment.

LINE SEGMENT. A special case of the curves between two points. It may be represented by a string stretched tautly between its two endpoints.

M

MATCH. Two sets match each other if their members can be put in one-to-one correspondence.

MEASURE. A number assigned to a geometric figure indicating its size with respect to a specific unit.

MEMBER OF A SET. An object or element in a set.

METRIC SYSTEM. A decimal system of measure with the meter as the standard unit of length.

MULTIPLE OF A WHOLE NUMBER. A product of that number and any whole number.

N

NEGATIVE RATIONAL NUMBER. The opposite of a positive rational number. (See OPPOSITE NUMBERS.)

NON-NEGATIVE RATIONAL NUMBER. All the positive rational numbers and zero.

NUMBER.
See
Whole number
Counting number
Rational number
Negative rational number
Irrational number
Real number .

NUMBER LINE. A model to show numbers and their order. The model is used first for the whole numbers. The markings and names are extended as the number system is extended until finally a 1-1 correspondence is set up between all the points of the line and all the real numbers.

NUMERAL. A name or symbol used for a number.

NUMERATION SYSTEM. A numeral system for naming numbers.

NUMBER SENTENCE. A mathematical sentence stating a relationship between numbers.

O

ONE-TO-ONE CORRESPONDENCE. A pairing between two sets A, B, which associates with each member of A a single member of B, and with each member of B a single member of A.

OPEN SENTENCE. A sentence with one or more symbols that may be replaced by the elements of a given set.

OPERATION. A (binary) operation is an association of an ordered pair of numbers with a third number.

OPPOSITE NUMBERS. A pair of numbers whose sum is 0.

ORDER. A property of a set of numbers which permits one to say when *a* and *b* are in the set whether *a* is "less than," "greater than," or "equal to" *b*.

ORDERED PAIR. An ordered pair of objects is a set of two objects in which one of them is specified as being first.

P

PAIRING. A correspondence between an element of one set and an element of another set.

PARALLEL LINES. Lines in the same plane which do not intersect.

PARALLEL PLANES. Planes that do not intersect.

PARALLELOGRAM. A quadrilateral whose opposite sides are parallel.

PERCENT. Means "per hundred," as 3 per hundred or 3 percent.

PERIMETER. The total length of a closed curve.

PLACE VALUES. The values given to the different positions in a numeral.

PLACE VALUE NUMERATION SYSTEM. A numeration system which uses the position or place in the numeral to indicate the value of the digit in that place.

PLANE. A particular set of points. It can be thought of as the extension of a flat surface such as a table. Usually an undefined term in geometry.

PLANE CURVE. A plane curve is a curve all points of which lie in a plane.

PLANE CLOSED REGION. The interior of any simple closed plane curve together with the curve.

POINT. An undefined term. It may be thought of as an exact location in space.

POLYGON. A simple closed curve in a plane which is the union of three or more line segments.

POSITIVE RATIONAL NUMBER. Any number that can be expressed as $\frac{a}{b}$ where a is a whole number and b is a counting number.

POSTULATE. A statement which is accepted without proof.

PRIME NUMBER. Any whole number that has exactly two different factors (namely itself and 1).

PRISM. A surface consisting of two congruent polygonal regions as bases and plane regions bounded by parallelograms as lateral faces.

PROPORTION. A statement of equality between two ratios.

PYRAMID. A surface which is a set of points consisting of a polygonal region called the base, a point called the vertex not in the same plane as the base, and all the triangular regions determined by the vertex and the sides of the base.

R

RADIUS OF CIRCLE. A line segment with one endpoint the center of the circle and the other endpoint on the circle.

RATIO. A relationship $a:b$ between an ordered pair of numbers a and b where $b \neq 0$. The ratio may be also expressed by the fraction $\frac{a}{b}$.

RATIONAL NUMBER. Any number which can be written in the form $\frac{a}{b}$ where a is an integer and b is a counting number.

RAY. The union of a point A and all those points of the line $\overleftrightarrow{AB}$ on the same side of A as B.

REAL NUMBERS. The union of the set of rational numbers and the set of irrational numbers.

RECIPROCALS. Any pair of numbers whose product is 1.

REGION. See PLANE REGION.

REGROUPING. A word used to replace the words "carrying" and "borrowing."

RIGHT RECTANGULAR PRISM. A right prism whose base is a rectangle.

S

SEGMENT. See LINE SEGMENT.

SEPARATE. To divide a given set of points such as a line, plane, sphere, space, etc. into disjoint subsets by use of another subset such as a point, line, circle, plane, etc.

SET. A set is any collection of things listed or specified well enough so that one can say exactly whether a certain thing does or does not belong to it.

SIMPLE CLOSED CURVE. A plane closed curve which does not intersect itself.

SIMILAR. A relationship between two geometric figures which have the same shape but not necessarily the same size.

SKEW. Two lines which do not intersect and are not parallel.

SOLUTION SET. The set of all numbers which make an open number sentence true.

SPACE. The set of all points.

SUBSET. Given two sets A and B, B is a subset of A if every member of B is also a member of A.

T

TRIANGLE. A polygon with three sides.

U

UNION OF TWO SETS. The union of two sets is the set of all elements that are in at least one of the given sets.

UNIQUE. An adjective meaning one and only one.

V

VERTEX (pl. VERTICES).

of an angle: the common endpoint of its two rays.

of a polygon: the common endpoint of two segments.

of a prism or pyramid: the common endpoint of three or more edges.

VOLUME. A measurement in terms of a specified unit which is assigned to a solid region. Note that both number and unit must be given, as 3 cubic feet.

W

WHOLE NUMBER. The counting numbers and the number 0: $\{0, 1, 2, 3, 4, \ldots\}$.

Z

ZERO. The number associated with the empty set.

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